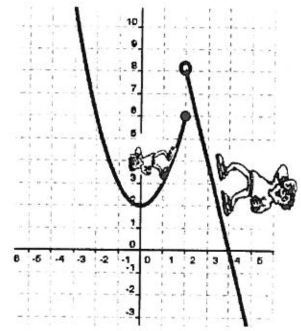
Name: Answer key Per: ______ Date: __________

Serafino ▪ Precalculus S2

13R Limits Recap & Review



Limits: ***Where do you think you're going?***

A limit is, essentially, the function's predicted destination based on its current path.

The "Left–hand limit" is the predicted destination coming from the left
"The limit of f(x) as x approaches c from the left" $lim_{x \to c^-} f(x) =$

The "Right–hand limit" is the predicted destination coming from the right.
"The limit of f(x) as x approaches c from the right" $lim_{x \to c^+} f(x) =$

"The Limit" of a function is the predicted destination when an x–value is approached from BOTH directions. *"The limit of f(x) as x approaches c"* $lim_{x \to c} f(x) =$

When do limits exist? When the limit from both sides is the same and it's a NUMBER.

Limits DO exist when the function is continuous at the approached x–value.

Limits DO exist when the approached x–value is a removable discontinuity: The left limit = right limit.

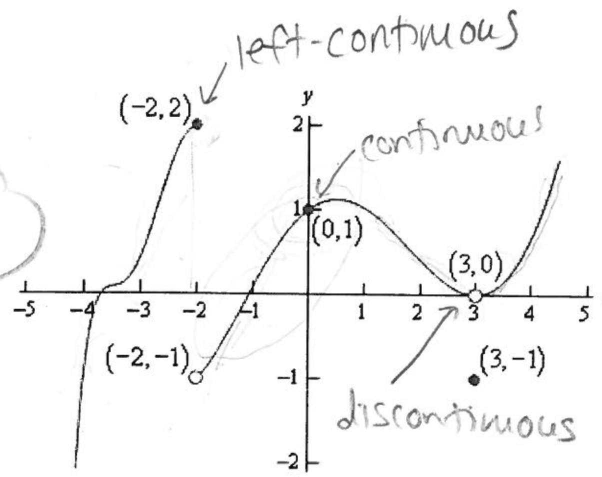
When do limits NOT exist?

Limits DO NOT exist at a jump discontintuity; when the left limit ≠ right limit.

Limits DO NOT exist when the function approaches ∞ or −∞; as end behavior or at a vertical asymptote.

1. Use the graph to evaluate the limits:

Left–hand Limit	Right–hand Limit	**The Limit**
$\lim_{x \to -2^-} f(x) = 2$	$\lim_{x \to -2^+} f(x) = -1$	$\lim_{x \to -2} f(x) =$ DNE (L≠R)
$\lim_{x \to 0^-} f(x) = 1$	$\lim_{x \to 0^+} f(x) = 1$	$\lim_{x \to 0} f(x) = 1$
$\lim_{x \to 3^-} f(x) = 0$	$\lim_{x \to 3^+} f(x) = 0$	$\lim_{x \to 3} f(x) = 0$



**Note: We don't care what f(c) actually is. Limits are about APPROACHING destinations...
It's about the predicted location the closer and closer we get to the x–value.*

Continuity: ***"Will you get there uninterrupted?"***

A function is "continuous" if there are no breaks. If you can draw the function throughout its domain without lifting your pencil off the page. Many functions are continuous (lines, parabolas, sinusoids, etc)... but many are not (rationals, radicals, tangent, secant, etc)

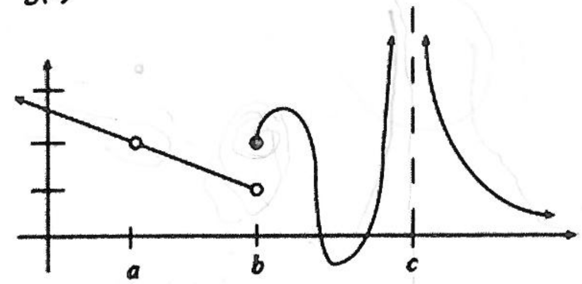
Interval Continuity: If a function is not continuous, then it has discontinuities. But the function is still continuous on certain intervals of the domain.

2. Name five intervals on which f(x), above is continuous:

$x < -2$, $-2 < x < 3$, $x > 3$,
$0 < x < 2$, $-4 < x < 3$

Types of Discontinuities:

$g(x) =$



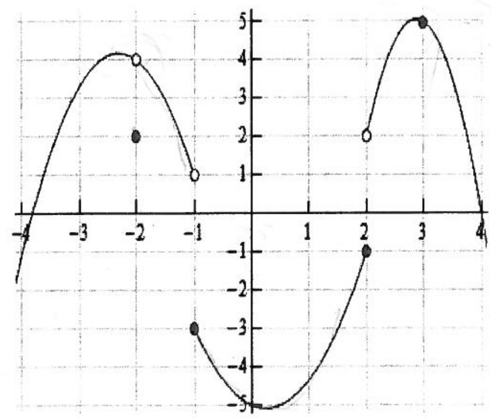
x → a Removeable: a hole; a point is removed from the curve
The limit exists; you'll get an actual number

x → b Jump: the curve picks up at a different height
The limit DNE, but (left ≠ right) or (jump)

x → c Infinite: Vertical asymptote
The limit DNE, but it could denote ∞, $-\infty$

3. Fill in the table from the graph. If the limit exists, then write the general limit statement. If not, write DNE.

Left–hand Limit	Right–hand Limit	The Limit
$\lim_{x\to-2^-} f(x) =$ 4	$\lim_{x\to-2^+} f(x) =$ 4	$\lim_{x\to-2} f(x) = 4$
$\lim_{x\to-1^-} f(x) =$ 1	$\lim_{x\to-1^+} f(x) =$ −3	$\lim_{x\to-1} f(x) =$ DNE (L≠R)
$\lim_{x\to2^-} f(x) =$ −1	$\lim_{x\to2^+} f(x) =$ 2	$\lim_{x\to2} f(x) =$ DNE (L≠R)
$\lim_{x\to3^-} f(x) =$ 5	$\lim_{x\to3^+} f(x) =$ 5	$\lim_{x\to3} f(x) = 5$



Continuity at a Point: ***... do you END UP getting where you thought you'd go?***
If limits are about the predicted destination of a journey towards a point...
Continuity at a point is about reaching the predicted destination. A function is...

"Left–continuous" at a point if the path from the left leads to the predicted value.
"Right–continuous" at a point if the path from the left leads to the predicted value.
"Continuous" at a point if the left and right path both lead to AND get there.

Evaluate
f(–2)= 2
f(–1)= −3
f(2) = −1
f(3)= 5

4. Evaluate the function and name the x–value that at which f(x) is...

a. Left–continuous (only) x = 2

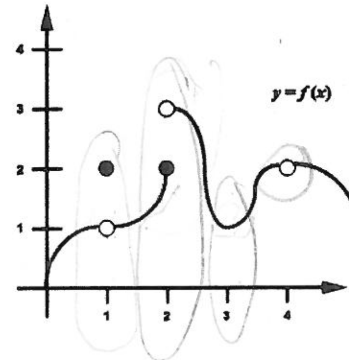
b. Right–continuous (only) x = −1

d. Discontinuous, but the Limit exists x = −2

d. Continuous (name at least 3) x = 3, x = 1, x = c, x = −1.2, etc

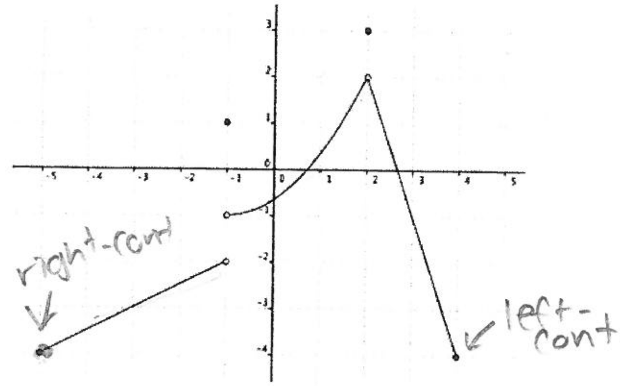
5. For the graph, classify the type of Continuity and/or Discontinuity & the Limit at the given values of x.

	Continuity (L/R)	Discontinuity	Limit
x = 1	none	removable	1
x = 2	left	jump	DNE
x = 3	continuous	none	1
x = 4	none	removable	2



Writing one–handed and general limit statements from a graph:
★ If the limit DNE because of a jump discontinuity (the left–hand limit doesn't equal the right–hand limit) write DNE, but also state why: (left ≠ right)… (because DNE can apply to other scenarios)

6. Writing Limits statements from a graph and evaluate.



a. Write a one–handed limit statement for f(x) at x = –5

$\lim_{x \to -5^+} f(x) = -4$

b. Write a one–handed limit statement for f(x) at x = 4

$\lim_{x \to 4^-} f(x) = -4$

c. The limit as x approaches –1 from the left.

$\lim_{x \to -1^-} f(x) = -2$

d. The limit as x approaches –1 from the right.

$\lim_{x \to -1^+} f(x) = -1$

e. The limit as x approaches –1.

$\lim_{x \to -1} f(x) = \text{DNE}$ (L≠R)

f. The limit as x approaches 2 from the left.

$\lim_{x \to 2^-} f(x) = 2$

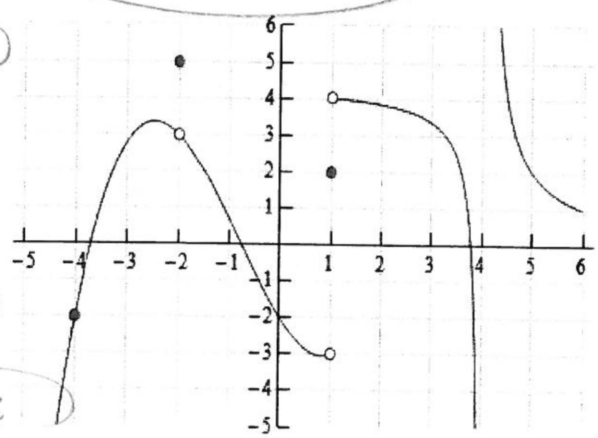
g. The limit as x approaches 2 from the right

$\lim_{x \to 2^+} f(x) = 2$

h. The limit as x approaches 2.

$\lim_{x \to 2} f(x) = 2$

7. Use the graph to evaluate. If the limit DNE, state why.



$\lim_{x \to -4^-} f(x) = -2$ $\quad \lim_{x \to -4^+} f(x) = -2$ $\quad \lim_{x \to -4} f(x) = -2$

$\lim_{x \to -2^-} f(x) = 3$ $\quad \lim_{x \to -2^+} f(x) = 3$ $\quad \lim_{x \to -2} f(x) = 3$

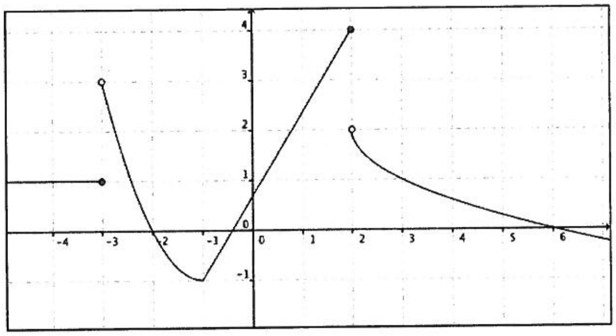
$\lim_{x \to 0^-} f(x) = -2$ $\quad \lim_{x \to 0^+} f(x) = -2$ $\quad \lim_{x \to 0} f(x) = -2$

$\lim_{x \to 1^-} f(x) = -3$ $\quad \lim_{x \to 1^+} f(x) = 4$ $\quad \lim_{x \to 1} f(x) = \text{DNE}$ (L≠R)

★ When the function is going to ∞ or –∞, the limit technically DNE (how can you limit something to an infinite number??) So write DNE, but also write (∞) or (–∞) or (VA) depending on what's happening.

$\lim_{x \to 4^-} f(x) =$ DNE $(-\infty)$ $\quad \lim_{x \to 4^+} f(x) =$ DNE (∞) $\quad \lim_{x \to 4} f(x) =$ DNE (VA) $\quad \lim_{x \to -\infty} f(x) =$ DNE $(-\infty)$ $\quad \lim_{x \to \infty} f(x) = 0$

8. Writing Equations from the graph. Put each piece in order from left to right in the domain.



$$f(x) = \begin{cases} 1, & x \le -3 \\ (x+1)^2 - 1, & -3 < x \le -1 \\ \frac{5}{3}(x+1) - 1, & -1 \le x \le 2 \\ -\sqrt{(x-2)} + 2, & x > 2 \end{cases}$$

Use your piecewise function to evaluate the limits:

b. $\lim\limits_{x\to 0} f(x) = \boxed{\frac{2}{3}}$ $\qquad \frac{5}{3}(0+1) - \frac{3}{3}$

$\lim\limits_{x\to 3/2} f(x) = \boxed{19/6}$ $\qquad \frac{5}{2}\,\frac{5}{3}\left(\frac{3}{2}+\frac{2}{2}\right) - \frac{6}{6}$

$\lim\limits_{x\to 20} f(x) = \boxed{2 - 3\sqrt{2}}$ $\qquad -\sqrt{18} + 2$ $\qquad -3\sqrt{2} + 2$

$\lim\limits_{x\to -\infty} f(x) = \boxed{1}$

$\lim\limits_{x\to \infty} f(x) = \boxed{\text{DNE } (-\infty)}$

c. Evaluate the your function to show that:

left $\rightarrow$ • $\leftarrow$ right

f(x) is continuous at x = –1

left $\lim\limits_{x\to -1^-} f(x) = -1$ $\qquad f(-1) = -1$

right $\lim\limits_{x\to -1^+} f(x) = -1$ ✓

✓ ✓ ✗

$\rightarrow$ • $\leftarrow$

f(x) is only left–continuous x = 2

left $\lim\limits_{x\to 2^-} f(x) = 4$ $\qquad f(2) = 4$

right $\lim\limits_{x\to 2^+} f(x) = 2$

9. Classify the continuity/discontinuity at x = 3? Note which "piece" is approaching from the left vs. right.

$$f(x) = \begin{cases} 2 - 5x & x \le 3 \\ 4x + 2 & x > 3 \end{cases}$$

$\lim\limits_{x\to 3^-} f(x) = 2 - 5(3) = -13$

$\lim\limits_{x\to 3^+} f(x) = 4(3) + 2 = 15$

↑ is left-continuous @ x=3

$f(3) = 2 - 5(3) = -13$

$\lim\limits_{x\to 3} f(x) =$ DNE $(L \ne R)$

10. Find the value of k that makes each of the following piecewise functions continuous:

$$f(x) = \begin{cases} x^2 + k & x < 4 \\ 5x - 1 & x \ge 4 \end{cases}$$

$5(4) - 1 = 19$

$4^2 + k = 19$

$\boxed{k = 3}$

$$h(x) = \begin{cases} 2x^3 + k & x \le 5 \\ -\frac{1}{2}(x+5)^2 & x > 5 \end{cases}$$

$-\frac{1}{2}(5+5)^2 = -50$

$2(5)^3 + k = -50$

$250 + k = -50$

$\boxed{k = -300}$

$$g(x) = \begin{cases} k & x \le 0 \\ -x^2 + k & 0 \le x \le 2 \\ x - 1 & 2 \le x \le 4 \end{cases}$$

$2 - 1 = 1$

$-(2)^2 + k = 1$

$-4 + k = 1,\ k = 5$

$\boxed{k = 5}$

11. Evaluate the left–and–right hand limits of the following piecewise functions.

$$g(x)=\begin{cases}\sqrt{5-x}, & x<-4\\ x^2-5, & -4\le x<2\\ x-3, & x\ge 2\end{cases}$$

$\lim_{x\to 2^-} g(x) =$ 4-5 = -1

$\lim_{x\to -4^+} g(x) =$ 16-5 = 11

$g(2) =$ 2-3 -1

$\lim_{x\to -4^-} g(x) =$ $\sqrt{5-4}$ = 3

$\lim_{x\to 2^+} g(x) =$ 2-3 -1

$\lim_{x\to 2} g(x) =$ -1

$\lim_{x\to -4} g(x) =$ DNE (L≠R)

$g(-4) =$ 16-5 = 11

$$h(x)=\begin{cases}-|x|, & x\le -5\\ 20-x^2, & -5<x\le 3\\ 4x-1, & x>3\end{cases}$$

$\lim_{x\to -5^+} h(x) =$ $20-(-5)^2$ = -5
20-25

$\lim_{x\to -5} h(x) =$ -5

$h(3) =$ 20-9 11

$\lim_{x\to -5^-} h(x) =$ $-|-5|$ -5

$\lim_{x\to 3^+} h(x) =$ 4(3)-1 11

$\lim_{x\to 3} h(x) =$ 20-9 11

$h(-5) =$ $-|-5|$ = -5

$\lim_{x\to 3^-} h(x) =$ 20-9 = 11

$$w(\theta)=\begin{cases}\sin\theta, & \theta\le\pi\\ \cos\theta, & \pi<\theta<2\pi\\ \tan\theta, & \theta>2\pi\end{cases}$$

$\lim_{x\to \pi^-} w(\theta) =$ sin(π) = 0

$w(\pi) =$ sin(π) = 0

$\lim_{x\to \pi^+} w(\theta) =$ cos(π) = -1

$\lim_{x\to 2\pi^-} w(\theta) =$ cos(2π) = 1

$\lim_{x\to \pi} w(\theta) =$ DNE (L≠R)

$\lim_{x\to 2\pi^+} w(\theta) =$ tan 2π = 0

$\lim_{x\to 2\pi} w(\theta) =$ DNE (L≠R)

$w(2\pi) =$ DNE (undefined)

What type of discontinuities, if any, occur at the "switch–over" values of each piecewise function?

x = 2 is continuous
x = -4 is right-continuous (jump)

x = 3 is cont.
x = -5 is cont.

x = π is left-cont.
x = 2π is discontinuous

Evaluating a limits as x approaches some sort of infinity? Use the END BEHAVIOR!
You can memorize the "3 rules" – or just realize that it's really just ONE rule that has three consequences. Just divide the leading terms of the top polynomial by the bottom. The resulting term is what the ends of the function will do.

$\lim_{x\to\infty}\frac{5}{x^2}$ = 0

$\lim_{x\to\infty}\frac{10^{10}}{x^{1/2}}$ = 0

$\lim_{x\to\infty}\frac{3x+2}{x^4}$ = 0

$\lim_{x\to\infty}\frac{x^2-4x+7}{x^3+5x}$ = 0

$\lim_{x\to\infty}\left(\frac{7x^2-3x+4}{2x^2-5x+3}\right)$ = 7/2

$\lim_{x\to\infty}\left(\frac{5x-4}{8-3x^2}\right)$ = 0

$\lim_{x\to\infty}\left(\frac{(x-2)^3}{(x+2)^2}\right)$ = DNE (∞)

$\lim_{x\to-\infty}\frac{4}{x-3}$ = 0

$\lim_{x\to-\infty}\frac{x^3-3x+5}{x-8}$ = DNE (∞)

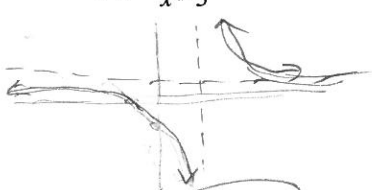
x^2

$\lim_{x\to-\infty}\frac{x^3-3x+5}{8-x}$ = DNE (-∞)

$\lim_{x\to\infty}\frac{5x^2+6x-4}{4x^2-3x+8}$ 5/4

12. **Using sketches of rational functions to determine one–sided limits as x → VAs:**

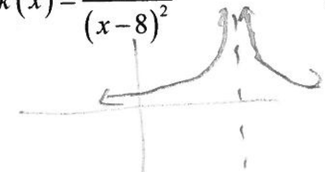
a. $f(x)=\dfrac{x+2}{x-3}$



$\lim_{x\to 3^-} f(x)$ DNE $(-\infty)$

$\lim_{x\to 3^+} f(x)$. DNE (∞)

b. $k(x)=\dfrac{1}{(x-8)^2}$



$\lim_{x\to 8^-} k(x)$ DNE (∞)

$\lim_{x\to 8^+} k(x)$ DNE (∞)

c. $g(x)=\dfrac{x+3}{(x-4)(x+6)}$ $\quad \frac{3}{-24}\ \frac{1}{6}$

$\lim_{x\to -6^-} g(x)$ DNE $(-\infty)$

$\lim_{x\to -6^+} g(x)$ DNE (∞)

$\lim_{x\to 4^-} g(x)$ DNE $(-\infty)$

$\lim_{x\to 4^+} g(x)$ DNE (∞)

$\lim_{x\to -\infty} g(x) = 0$

$\lim_{x\to \infty} g(x) = 0$

Evaluating Limits: Recall the Calc–Help videos. There are 3 methods: substitution, factorization & conjugation. If the limit does not exist, state "DNE" and why.

$\lim_{x\to -3}(x^3) = -27$

$\lim_{t\to -2}\left(\dfrac{t^2+4}{t-4}\right) = -4/3$

$\dfrac{4+4}{-6} = -8/6$

$\lim_{x\to 3}(x(x+2)(x+3))$

$3 \cdot 5 \cdot 6$

$3(3+2)(3+3)$

$= 90$

$\lim_{x\to -1}\left(\dfrac{x+3}{x^2-6x+8}\right) = \dfrac{2}{15}$

$2/(-5)(-3)$

$(x-4)(x-2)$

$\lim_{x\to 2}\left(\sqrt[5]{3x^2-2x}\right)$ $\quad \sqrt[5]{8}$

$\sqrt{3(4)-4}$

$\lim_{x\to 2}\left(\dfrac{x-2}{x-2}\right) = 1$

$\lim_{x\to -4}\left(\dfrac{x^2-16}{x+4}\right) = \dfrac{(x-4)\cancel{(x+4)}}{\cancel{(x+4)}}$

-8

$\lim_{x\to 3}\dfrac{x^2-4x+3}{x^2+x-12} = \dfrac{\cancel{(x-3)}(x-1)}{(x+4)\cancel{(x-3)}}$

$\frac{2}{7}$ $\quad \dfrac{2}{7}$

$\lim_{x\to 2}\dfrac{x^3-8}{x-2}$ $\quad \dfrac{\cancel{(x-2)}(x^2+2x+4)}{\cancel{(x-2)}}$

$4+4+4$

$= 12$

$\lim_{x\to 2}\dfrac{x^2-x+5}{x-2}$ DNE (VA)

$\lim_{x\to 1}\left(\dfrac{1}{x-1}-\dfrac{2}{x^2-1}\right)$

$\dfrac{1(x+1)}{x-1(x+1)} - \dfrac{2}{x^2-1}$

$x+1-2$

$\dfrac{\cancel{x-1}\quad 1}{\cancel{(x-1)}(x+1)}$

$= \boxed{\dfrac{1}{2}}$

$\lim_{h\to 5}\dfrac{h-5}{\sqrt{h+4}-3}$ $\quad \dfrac{\sqrt{h+4}-3}{\sqrt{h+4}-3}$

$h+4-9$

$\boxed{=0}$

$\lim_{x\to 4}\dfrac{\sqrt{x}-2}{x-4}$ $\quad \dfrac{(\sqrt{x}+2)}{\sqrt{x}+2}$

$\dfrac{\cancel{(x-4)}\quad 1}{\cancel{(x-4)}(\sqrt{x}+2)}$

$\dfrac{1}{2+2}$

$\dfrac{1}{4}$

$\lim_{x\to 0}\dfrac{\sqrt{x+2}-\sqrt{2}}{x}$ $\quad \dfrac{\sqrt{x+2}+\sqrt{2}}{\sqrt{x+2}+\sqrt{2}}$

$\dfrac{\cancel{x}+\cancel{2}-\cancel{2}\quad 1}{\cancel{x}\quad \sqrt{x+2}+\sqrt{2}}$

$\dfrac{1}{\sqrt{2}\sqrt{2}}$ $\quad \dfrac{1}{2\sqrt{2}}\dfrac{\sqrt{2}}{\sqrt{2}}$

$\boxed{\dfrac{\sqrt{2}}{2}}$

$\lim_{x\to 1}\dfrac{x-1}{\sqrt{x}-1}$ $\quad \dfrac{\sqrt{x}+1}{\sqrt{x}+1}$

$\cancel{x-1}$

$\boxed{=2}$