# SECTION 7.1 THE LAW OF SINES

There are many relationships that exist between the sides and angles in a triangle. One such relationship is called the *law of sines,* which states that the ratio of the sine of an angle to the length of the side opposite that angle is constant in any triangle. Here it is stated in symbols:

*Law of Sines*

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

or, equivalently,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

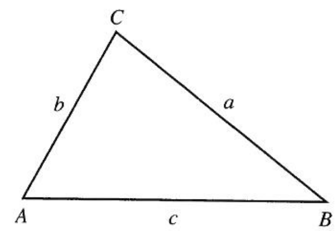


**Figure 1**

## PROOF

The altitude $h$ of the triangle in Figure 2 can be written in terms of $\sin A$ or $\sin B$ depending on which of the two right triangles we are referring to:

$$\sin A = \frac{h}{b} \qquad \sin B = \frac{h}{a}$$

$$h = b \sin A \qquad h = a \sin B$$

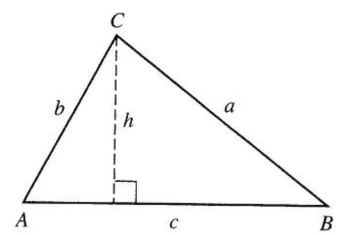


**Figure 2**

Since $h$ is equal to itself, we have

$$h = h$$

$$b \sin A = a \sin B$$

$$\frac{b \sin A}{ab} = \frac{a \sin B}{ab} \qquad \text{Divide both sides by } ab$$

$$\frac{\sin A}{a} = \frac{\sin B}{b} \qquad \text{Divide out common factors}$$

If we do the same kind of thing with the altitude that extends from $A$, we will have the third ratio in the law of sines, $\frac{\sin C}{c}$, equal to the two ratios above.

Note that the derivation of the law of sines will proceed in the same manner if triangle $ABC$ contains an obtuse angle, as in Figure 3.

In triangle $BDC$ we have

$$\sin (180° - B) = \frac{h}{a}$$

but,

$$\begin{aligned}\sin(180° - B) &= \sin 180° \cos B - \cos 180° \sin B \\ &= (0)\cos B - (-1)\sin B \\ &= \sin B\end{aligned}$$

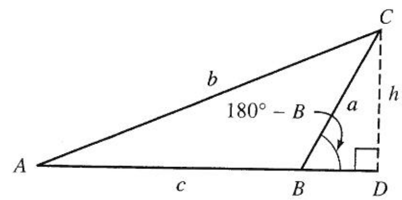


**Figure 3**

So, $\sin B = h/a$, which is the result we obtained previously. Using triangle $ADC$, we have $\sin A = h/b$. As you can see, these are the same two expressions we began with when deriving the law of sines for the acute triangle in Figure 2. From this point on, the derivation would match our previous derivation. ■

We can use the law of sines to find missing parts of triangles for which we are given two angles and a side.

## TWO ANGLES AND ONE SIDE

In our first example, we are given two angles and the side opposite one of them. (You may recall that in geometry these were the parts we needed equal in two triangles in order to prove them congruent using the AAS Theorem.)

**EXAMPLE 1** In triangle ABC, $A = 30°$, $B = 70°$, and $a = 8.0$ cm. Find the length of side $c$.

**SOLUTION** We begin by drawing a picture of triangle $ABC$ (it does not have to be accurate) and labeling it with the information we have been given.

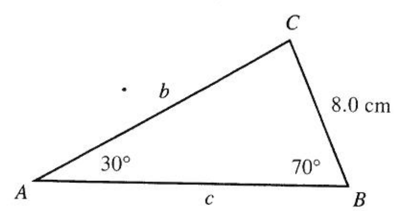


**Figure 4**

When we use the law of sines, we must have one of the ratios given to us. In this case, since we are given $a$ and $A$, we have the ratio $\dfrac{a}{\sin A}$. To solve for $c$, we need first to find angle $C$. Since the sum of the angles in any triangle is 180°, we have

$$\begin{aligned}C &= 180° - (A + B) \\ &= 180° - (30° + 70°) \\ &= 80°\end{aligned}$$

To find side $c$, we use the following two ratios given in the law of sines.

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

To solve for $c$, we multiply both sides by $\sin C$ and then substitute.

| | |
|---|---|
| $c = \dfrac{a \sin C}{\sin A}$ | Multiply both sides by $\sin C$ |
| $= \dfrac{8.0 \sin 80°}{\sin 30°}$ | Substitute in known values |
| $= \dfrac{8.0(0.9848)}{0.5000}$ | Calculator |
| $= 16$ cm | Rounded to the nearest integer |

*NOTE* The equal sign in the third line above should actually be replaced by the *approximately equal to* symbol, $\approx$, since the decimal 0.9848 is an approximation to sin 80°. (Remember, most of the trigonometric functions are irrational numbers.) In this chapter, we will use an equal sign in the solutions to all of our examples, even when the $\approx$ symbol would be more appropriate, in order to make the examples a little easier to follow.

In our next example, we are given two angles and the side included between them (ASA) and are asked to find all the missing parts.

**EXAMPLE 2** Solve triangle $ABC$ if $B = 34°$, $C = 82°$, and $a = 5.6$ cm.

**SOLUTION** We begin by finding angle $A$ so that we have one of the ratios in the law of sines completed.

*Angle A*

$$\begin{aligned} A &= 180° - (B + C) \\ &= 180° - (34° + 82°) \\ &= 64° \end{aligned}$$

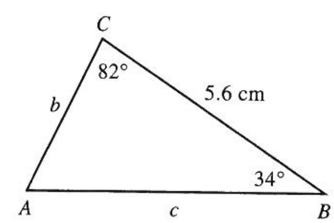


**Figure 5**

*Side b*

If $\dfrac{b}{\sin B} = \dfrac{a}{\sin A}$

then

| | |
|---|---|
| $b = \dfrac{a \sin B}{\sin A}$ | Multiply both sides by $\sin B$ |
| $= \dfrac{5.6 \sin 34°}{\sin 64°}$ | Substitute in known values |
| $= \dfrac{5.6(0.5592)}{0.8988}$ | Calculator |
| $= 3.5$ cm | To the nearest tenth |

*Side c*

$$\text{If} \qquad \frac{c}{\sin C} = \frac{a}{\sin A}$$

$$\text{then} \qquad c = \frac{a \sin C}{\sin A} \qquad \text{Multiply both sides by } \sin C$$

$$= \frac{5.6 \sin 82^\circ}{\sin 64^\circ} \qquad \text{Substitute in known values}$$

$$= \frac{5.6(0.9903)}{0.8988} \qquad \text{Calculator}$$

$$= 6.2 \text{ cm} \qquad \text{To the nearest tenth}$$

The law of sines, along with some fancy electronic equipment, was used to obtain the results of some of the field events in one of the recent Olympic Games.

Figure 6 is a diagram of a shot put ring. The shot is tossed (put) from the left and lands at $A$. A small electronic device is then placed at $A$ (there is usually a dent in the ground where the shot lands, so it is easy to find where to place the device). The device at $A$ sends a signal to a booth in the stands that gives the measures of angles $A$ and $B$. The distance $a$ is found ahead of time. To find the distance $x$, the law of sines is used.

$$\frac{x}{\sin B} = \frac{a}{\sin A}$$

$$\text{or} \qquad x = \frac{a \sin B}{\sin A}$$

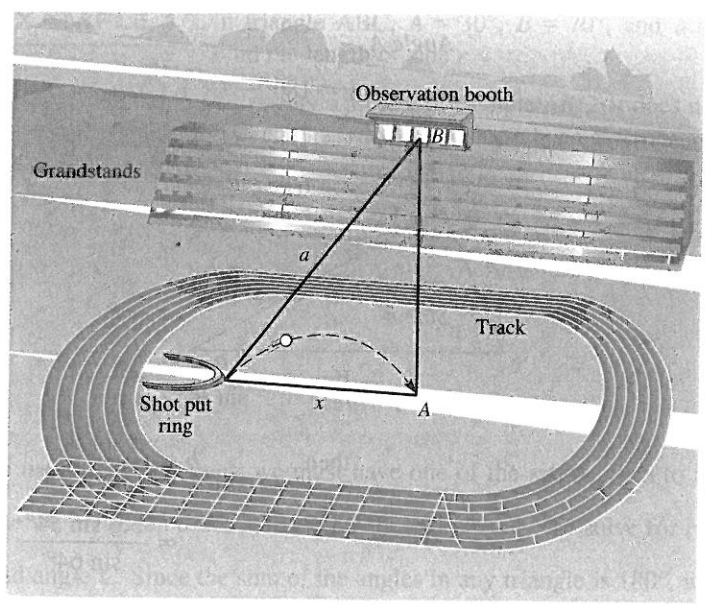


**Figure 6**

**EXAMPLE 3** Find $x$ in Figure 6 if $a = 562$ ft, $B = 5.7°$, and $A = 85.3°$.

**SOLUTION**

$$x = \frac{a \sin B}{\sin A}$$

$$= \frac{562 \sin 5.7°}{\sin 85.3°}$$

$$= 56.0 \text{ ft} \qquad \text{To three significant digits}$$

**EXAMPLE 4** A satellite is circling above the earth as shown in Figure 7. When the satellite is directly above point $B$, angle $A$ is 75.4°. If the distance between points $B$ and $D$ on the circumference of the earth is 910 miles and the radius of the earth is 3,960 miles, how far above the earth is the satellite?

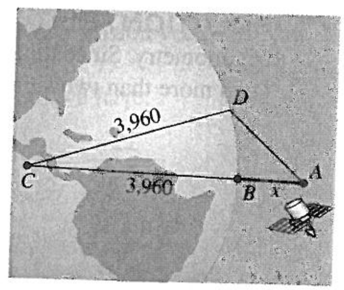


**Figure 7**

**SOLUTION** First we find the radian measure of central angle $C$ by dividing the arc length $BD$ by the radius of the earth. Multiplying this number by $180/\pi$ will give us the degree measure of angle $C$.

$$C = \underbrace{\frac{910}{3{,}960}}_{\text{Angle } C \text{ in radians}} \cdot \underbrace{\frac{180}{\ }}_{\text{Convert to degrees}} = 13.2°$$

Next we find angle $CDA$.

$$\angle CDA = 180° - (75.4° + 13.2°) = 91.4°$$

To find $x$, we use the law of sines.

$$\frac{x + 3{,}960}{\sin 91.4°} = \frac{3{,}960}{\sin 75.4°}$$

$$x + 3{,}960 = \frac{3{,}960 \sin 91.4°}{\sin 75.4°}$$

$$x = \frac{3{,}960 \sin 91.4°}{\sin 75.4°} - 3{,}960$$

$$x = 131 \text{ mi} \qquad \text{To three significant digits}$$

**EXAMPLE 5** A hot-air balloon is flying over a dry lake when the wind stops blowing. The balloon comes to a stop 450 feet above the ground at point $D$ as shown in Figure 8. A jeep following the balloon runs out of gas at point $A$. The nearest service station is due north of the jeep at point $B$. The bearing of the balloon from the jeep at $A$ is N 13° E, while the bearing of the balloon from the service station at $B$ is S 19° E. If the angle of elevation of the balloon from $A$ is 12°, how far will the people in the jeep have to walk to reach the service station at point $B$?

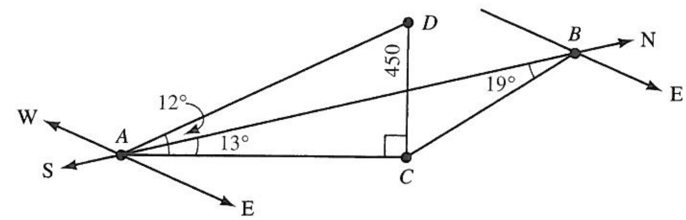


**Figure 8**

**SOLUTION** First we find the distance between $C$ and $A$ using right triangle trigonometry. Since this is an intermediate calculation, which we will use again, we keep more than two significant digits for $AC$.

$$\tan 12^\circ = \frac{450}{AC}$$

$$AC = \frac{450}{\tan 12^\circ}$$

$$= 2{,}117 \text{ ft}$$

Next we find angle $ACB$.

$$\angle ACB = 180^\circ - (13^\circ + 19^\circ)$$

$$= 148^\circ$$

Finally, we find $AB$ using the law of sines.

$$\frac{AB}{\sin 148^\circ} = \frac{2{,}117}{\sin 19^\circ}$$

$$AB = \frac{2{,}117 \sin 148^\circ}{\sin 19^\circ}$$

$$= 3{,}400 \text{ ft} \qquad \text{To two significant digits}$$

Since there are 5,280 feet in a mile, the people at $A$ will walk approximately 3,400/5,280 = 0.6 miles to get to the service station at $B$. ■

Our next example involves vectors. It is taken from the text *College Physics* by Miller and Schroeer, published by Saunders College Publishing.

**EXAMPLE 6** A traffic light weighing 22 pounds is suspended by two wires as shown in Figure 9. Find the magnitude of the tension in wire $AB$, and the magnitude of the tension in wire $AC$.

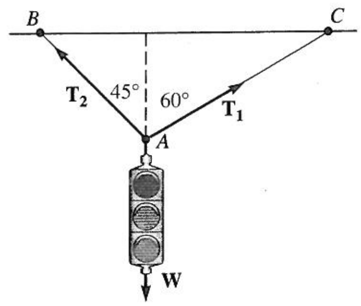


**Figure 9**

**SOLUTION** We assume that the traffic light is not moving and is therefore in the state of *static equilibrium*. When an object is in this state, the sum of the forces acting on the object must be 0. It is because of this fact that we can redraw the vectors from Figure 9 and be sure that they form a closed triangle. Figure 10 shows a convenient redrawing of the two tension vectors $\mathbf{T}_1$ and $\mathbf{T}_2$, and the vector $\mathbf{W}$ that is due to gravity.

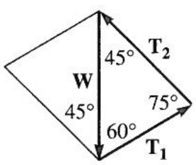


**Figure 10**

Using the law of sines we have:

$$\frac{|\mathbf{T}_1|}{\sin 45°} = \frac{22}{\sin 75°}$$

$$|\mathbf{T}_1| = \frac{22 \sin 45°}{\sin 75°}$$

$$= 16 \text{ lb} \qquad \text{To two significant figures}$$

$$\frac{|\mathbf{T}_2|}{\sin 60°} = \frac{22}{\sin 75°}$$

$$|\mathbf{T}_2| = \frac{22 \sin 60°}{\sin 75°}$$

$$= 20 \text{ lb} \qquad \text{To two significant figures}$$

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

**a.** What is the law of sines?

**b.** Find $\sin(180° - B)$.

**c.** Why is it always possible to find the third angle of any triangle once you are given the first two?

**d.** When an object is in the state of static equilibrium, what do we know about the forces acting upon that object?

## PROBLEM SET 7.1

Each problem that follows refers to triangle $ABC$.

1. If $A = 40°$, $B = 60°$, and $a = 12$ cm, find $b$.
2. If $A = 80°$, $B = 30°$, and $b = 14$ cm, find $a$.
3. If $B = 120°$, $C = 20°$, and $c = 28$ inches, find $b$.
4. If $B = 110°$, $C = 40°$, and $b = 18$ inches, find $c$.
5. If $A = 10°$, $C = 100°$, and $a = 24$ yd, find $c$.
6. If $A = 5°$, $C = 125°$, and $c = 510$ yd, find $a$.
7. If $A = 50°$, $B = 60°$, and $a = 36$ km, find $C$ and then find $c$.
8. If $B = 40°$, $C = 70°$, and $c = 42$ km, find $A$ and then find $a$.
9. If $A = 52°$, $B = 48°$, and $c = 14$ cm, find $C$ and then find $a$.
10. If $A = 33°$, $C = 82°$, and $b = 18$ cm, find $B$ and then find $c$.

The information below refers to triangle $ABC$. In each case, find all the missing parts.

11. $A = 42.5°$, $B = 71.4°$, $a = 215$ inches
12. $A = 110.4°$, $C = 21.8°$, $c = 246$ inches
13. $A = 46°$, $B = 95°$, $c = 6.8$ m
14. $B = 57°$, $C = 31°$, $a = 7.3$ m
15. $A = 43° 30'$, $C = 120° 30'$, $a = 3.48$ ft
16. $B = 14° 20'$, $C = 75° 40'$, $b = 2.72$ ft
17. $B = 13.4°$, $C = 24.8°$, $a = 315$ cm
18. $A = 105°$, $B = 45°$, $c = 630$ cm

19. In triangle $ABC$, $A = 30°$, $b = 20$ ft, and $a = 2$ ft. Show that it is impossible to solve this triangle by using the law of sines to find sin $B$.
20. In triangle $ABC$, $A = 40°$, $b = 20$ ft, and $a = 18$ ft. Use the law of sines to find sin $B$ and then give two possible values for $B$.

**Geometry** The circle in Figure 11 has a radius of $r$ and center at $C$. The distance from $A$ to $B$ is $x$, the distance from $A$ to $D$ is $y$, and the length of arc $BD$ is $s$. For Problems 21 through 24, redraw Figure 11, label it as indicated in each problem, and then solve the problem.

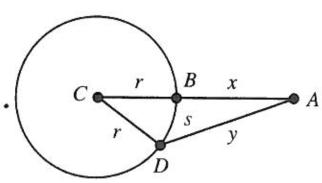


**Figure 11**

21. If $A = 31°$, $s = 11$, and $r = 12$, find $x$.
22. If $A = 26°$, $s = 22$, and $r = 20$, find $x$.
23. If $A = 45°$, $s = 18$, and $r = 15$, find $y$.
24. If $A = 55°$, $s = 21$, and $r = 22$, find $y$.

25. **Angle of Elevation** A man standing near a radio station antenna observes that the angle of elevation to the top of the antenna is 64°. He then walks 100 feet further away and observes that the angle of elevation to the top of the antenna is 46°. Find the height of the antenna to the nearest foot. (*Hint:* Find $x$ first.)

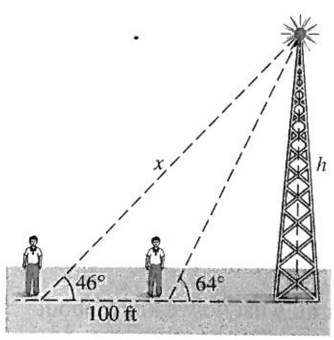


**Figure 12**

**26. Angle of Elevation** A person standing on the street looks up to the top of a building and finds that the angle of elevation is 38°. She then walks one block further away (440 feet) and finds that the angle of elevation to the top of the building is now 28°. How far away from the building is she when she makes her second observation? (See Figure 13.)

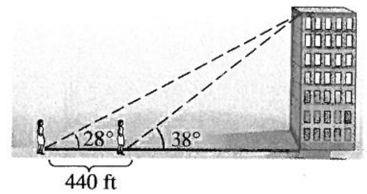


**Figure 13**

**27. Angle of Depression** A man is flying in a hot-air balloon in a straight line at a constant rate of 5 feet per second, while keeping it at a constant altitude. As he approaches the parking lot of a market, he notices that the angle of depression from his balloon to a friend's car in the parking lot is 35°. A minute and a half later, after flying directly over this friend's car, he looks back to see his friend getting into the car and observes the angle of depression to be 36°. At that time, what is the distance between him and his friend? (Give your answer to the nearest foot.)

**28. Angle of Elevation** A woman entering an outside glass elevator on the ground floor of a hotel glances up to the top of the building across the street and notices that the angle of elevation is 48°. She rides the elevator up three floors (60 feet) and finds that the angle of elevation to the top of the building across the street is 32°. How tall is the building across the street? (Give your answer to the nearest foot.)

**29. Angle of Elevation** From a point on the ground, a person notices that a 110-foot antenna on the top of a hill subtends an angle of 0.5°. If the angle of elevation to the bottom of the antenna is 35°, find the height of the hill. (See Figure 14.)

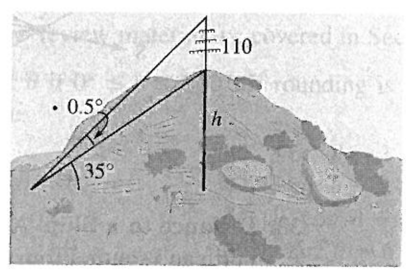


**Figure 14**

**30. Angle of Elevation** A 150-foot antenna is on top of a tall building. From a point on the ground, the angle of elevation to the top of the antenna is 28.5°, while the angle of elevation to the bottom of the antenna from the same point is 23.5°. How tall is the building?

**31. Height of a Tree** Figure 15 is a diagram that shows how Colleen estimates the height of a tree that is on the other side of a stream. She stands at point $A$ facing the tree and finds the angle of elevation from $A$ to the top of the tree to be 51°. Then she turns 105° and walks 25 feet to point $B$, where she measures the angle between her path and the base of the tree. She finds that angle to be 44°. Use this information to find the height of the tree.

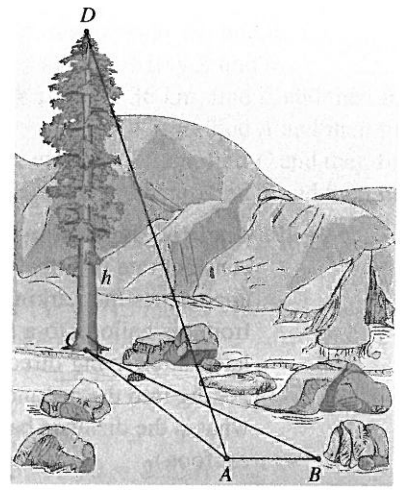


**Figure 15**

**32. Sea Rescue** A plane makes a forced landing at sea. The last radio signal received at station $C$ gives the bearing of the plane from $C$ as N 55.4° E at an altitude of 1,050 feet. An observer at $C$ sights the plane and gives $\angle DCB$ as 22.5°. How far will a rescue boat at $A$ have to travel to reach any survivors at $B$, if the bearing of $B$ from $A$ is S 56.4° E? (See Figure 16.)

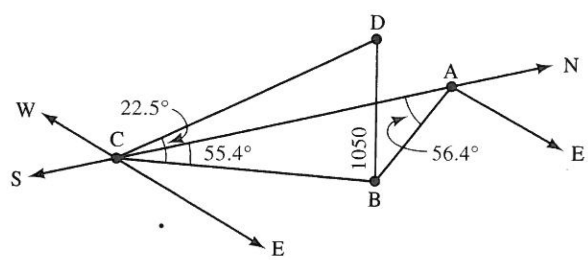


**Figure 16**

**33. Distance to a Ship** A ship is anchored off a long straight shoreline that runs north and south. From two observation points 18 miles apart on shore, the bearings of the ship are N 31° E and S 53° E. What is the distance from the ship to each of the observation points?

**34. Distance to a Rocket** Tom and Fred are 3.5 miles apart watching a rocket being launched from Vandenberg Air Force Base. Tom estimates the bearing of the rocket from his position to be S 75° W, while Fred estimates that the bearing of the rocket from his position is N 65° W. If Fred is due south of Tom, how far is each of them from the rocket?

**35. Force** A tightrope walker is standing still with one foot on the tightrope as shown in Figure 17. If the tightrope walker weighs 125 pounds, find the magnitudes of the tension in the rope toward each end of the rope.

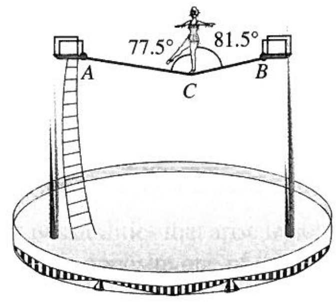


**Figure 17**

**36. Force** A tightrope walker weighing 145 pounds is standing still at the center of a tightrope that is 46.5 feet long. The weight of the walker causes the center of the tightrope to move down 14.5 inches. Find the magnitude of the tension in the tightrope.

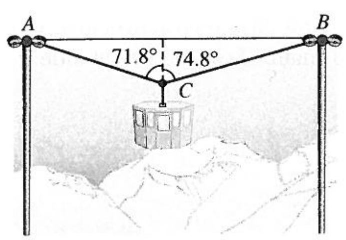


**Figure 18**

**37. Force** If you have ever ridden on a chair lift at a ski area and had it stop, you know that the chair will pull down on the cable, dropping you down to a lower height than when the chair is in motion. Figure 18 shows a gondola that is stopped. Find the magnitude of the tension in the cable toward each end of the cable if the total weight of the gondola and its occupants is 1,850 pounds.

**38. Force** A chair lift at a ski resort is stopped halfway between two poles that support the cable to which the chair is attached. The poles are 215 feet apart and the combined weight of the chair and the three people on the chair is 725 pounds. If the weight of the chair and the people riding it causes the chair to move to a position 15.8 feet below the horizontal line that connects the top of the two poles, find the tension in the cable.

## REVIEW PROBLEMS

The problems that follow review material we covered in Sections 3.1 and 6.1.

Solve each equation for $\theta$ if $0° \leq \theta < 360°$. If rounding is necessary, round to the nearest tenth of a degree.

**39.** $2 \sin \theta - \sqrt{2} = 0$

**40.** $5 \cos \theta - 3 = 0$

**41.** $\sin \theta \cos \theta - 2 \cos \theta = 0$

**42.** $3 \cos \theta - 2 \sin \theta \cos \theta = 0$

**43.** $2 \sin^2 \theta - 3 \sin \theta = -1$

**44.** $10 \cos^2 \theta + \cos \theta - 3 = 0$

**45.** $\cos^2 \theta - 4 \cos \theta + 2 = 0$

**46.** $2 \sin^2 \theta - 6 \sin \theta + 3 = 0$

Find all radian solutions to each equation using exact values only.

**47.** $(\sin x + 1)(2 \sin x - 1) = 0$

**48.** $2 \sin x \cos x - \sqrt{3} \sin x = 0$

Find $\theta$ to the nearest tenth of a degree if $0 \leq \theta < 360°$, and

**49.** $\sin \theta = 0.7380$

**50.** $\sin \theta = 0.7965$

**51.** $\sin \theta = 0.9668$

**52.** $\sin \theta = 0.2351$