Make a diagram of the unit circle with an angle θ in quadrant I and its supplement $180° - \theta$ in quadrant II. Label the point on the terminal side of θ and the unit circle with (x, y) and the point on the terminal side of $180° - \theta$ and the unit circle with $(-x, y)$. Use the diagram to show that

51. $\sin(180° - \theta) = \sin\theta$

52. $\cos(180° - \theta) = -\cos\theta$

53. Show that tangent is an odd function.

54. Show that cotangent is an odd function.

Prove each identity.

55. $\sin(-\theta)\cot(-\theta) = \cos\theta$

56. $\cos(-\theta)\tan\theta = \sin\theta$

57. $\sin(-\theta)\sec(-\theta)\cot(-\theta) = 1$

58. $\cos(-\theta)\csc(-\theta)\tan(-\theta) = 1$

59. $\csc\theta + \sin(-\theta) = \dfrac{\cos^2\theta}{\sin\theta}$

60. $\sec\theta - \cos(-\theta) = \dfrac{\sin^2\theta}{\cos\theta}$

61. Geometry Redraw the diagram in Figure 9 from this section and label the line segment that corresponds to $\sec t$.

62. Geometry Make a diagram similar to the diagram in Figure 9 from this section, but instead of labeling the point (1, 0) with B, label the point (0, 1) with B. Then place Q on the line OP and connect Q to B so that QB is perpendicular to the y-axis. Now, if $P(x, y)$ is t units from (1, 0), label the line segments that correspond to $\sin t$, $\cos t$, $\cot t$, and $\csc t$.

REVIEW PROBLEMS

The problems that follow review material we covered in Section 2.3.

Problems 63–70 refer to right triangle ABC in which $C = 90°$. Solve each triangle.

63. $A = 42°$, $c = 36$

64. $A = 58°$, $c = 17$

65. $B = 22°$, $b = 320$

66. $B = 48°$, $b = 270$

67. $a = 20.5$, $b = 31.4$

68. $a = 16.3$, $b = 20.8$

69. $a = 4.37$, $c = 6.21$

70. $a = 7.12$, $c = 8.44$

SECTION 3.4 ARC LENGTH AND AREA OF A SECTOR

In Chapter 2, we discussed some of the aspects of the first Ferris wheel, which was built by George Ferris in 1893. The first topic we will cover in this section is *arc length.* Our study of arc length will allow us to find the distance traveled by a rider on a Ferris wheel at any point during the ride.

In Section 3.2, we found that if a central angle θ, measured in radians, in a circle of radius r cuts off an arc of length s, then the relationship between s, r, and θ can be written as $\theta = \dfrac{s}{r}$. Figure 1 illustrates this.

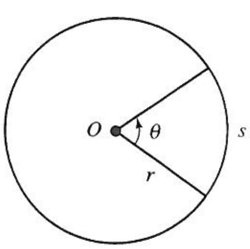


Figure 1

If we multiply both sides of this equation by r, we will obtain the equation that gives arc length s in terms of r and θ.

$$\theta = \frac{s}{r} \quad \text{Definition of radian measure}$$

$$r \cdot \theta = r \cdot \frac{s}{r} \quad \text{Multiply both sides by } r$$

$$r\theta = s$$

ARC LENGTH

If θ (in radians) is a central angle in a circle with radius r, then the length of the arc cut off by θ is given by

$$s = r\theta \qquad (\theta \text{ in radians})$$

EXAMPLE 1 Give the length of the arc cut off by a central angle of 2 radians in a circle of radius 4.3 inches.

SOLUTION We have $\theta = 2$ and $r = 4.3$ inches. Applying the formula $s = r\theta$ gives us

$$\begin{aligned} s &= r\theta \\ &= 4.3(2) \\ &= 8.6 \text{ inches} \end{aligned}$$

Figure 2 illustrates this example.

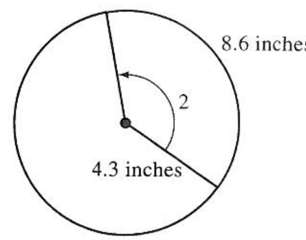
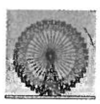


Figure 2



EXAMPLE 2 Figure 3 is a model of George Ferris's Ferris wheel. Recall that the diameter of the wheel is 250 feet, and θ is the central angle formed as a rider travels from his or her initial position P_0 to position P_1. Find the distance traveled by the rider if $\theta = 45°$ and if $\theta = 105°$.

SOLUTION The formula for arc length, $s = r\theta$, requires θ to be given in radians. Since θ is given in degrees, we must multiply it by $\pi/180$ to convert to radians. Also, since the diameter of the wheel is 250 feet, the radius is 125 feet.

For $\theta = 45°$:

$$\begin{aligned} s &= r\theta \\ &= 125(45)\left(\frac{\pi}{180}\right) \\ &= \frac{125\pi}{4} \text{ ft} \\ &\approx 98.2 \text{ ft} \end{aligned}$$

For $\theta = 105°$:

$$\begin{aligned} s &= r\theta \\ &= 125(105)\left(\frac{\pi}{180}\right) \\ &= \frac{875\pi}{12} \text{ ft} \\ &\approx 229.1 \text{ ft} \end{aligned}$$

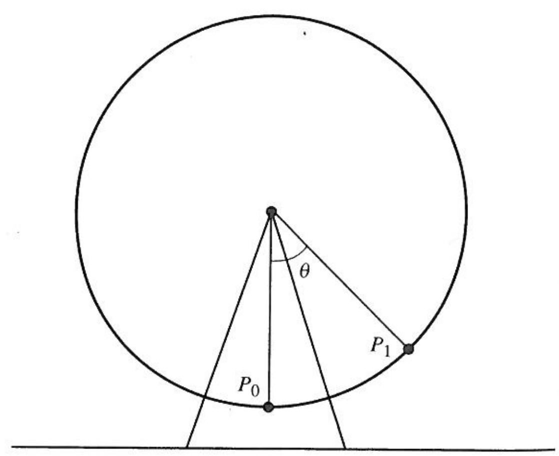


Figure 3

EXAMPLE 3 The minute hand of a clock is 1.2 centimeters long. To two significant digits, how far does the tip of the minute hand move in 20 minutes?

SOLUTION We have $r = 1.2$ cm. Since we are looking for s, we need to find θ. We can use a proportion to find θ. Since one complete rotation is 60 minutes and 2π radians, we say θ is to 2π as 20 minutes is to 60 minutes, or

$$\text{If} \quad \frac{\theta}{2\pi} = \frac{20}{60} \quad \text{then} \quad \theta = \frac{2\pi}{3}$$

Now we can find s.

$$\begin{aligned} s &= r\theta \\ &= 1.2\left(\frac{2\pi}{3}\right) \\ &= \frac{2.4\pi}{3} \\ &\approx 2.5 \text{ cm} \end{aligned}$$

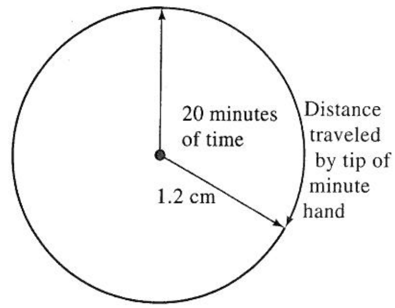


Figure 4

Figure 4 illustrates this example.

The tip of the minute hand will travel approximately 2.5 centimeters every 20 minutes.

If we are working with relatively small central angles in circles with large radii, we can use the length of the intercepted arc to approximate the length of the associated chord. For example, Figure 5 shows a central angle of 1° in a circle of radius 1,800 feet, along with the arc and chord cut off by 1°. (Figure 5 is not drawn to scale.)

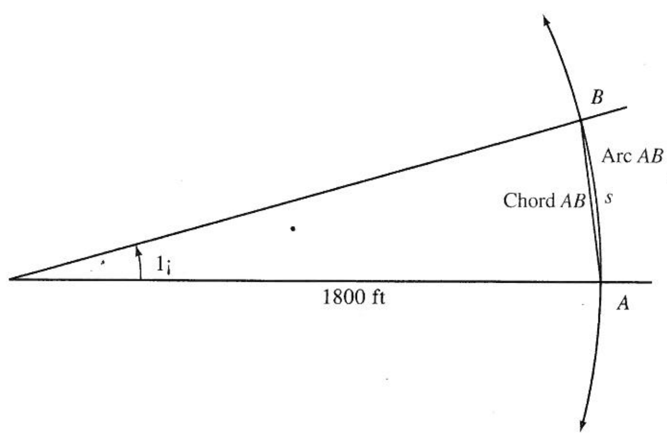


Figure 5

To find the length of arc AB, we convert θ to radians by multiplying by $\pi/180$. Then we apply the formula $s = r\theta$.

$$s = r\theta = 1{,}800(1)\left(\frac{\pi}{180}\right) = 10\pi \approx 31.4 \text{ ft}$$

If we had carried out the calculation of arc AB to six significant digits, we would have obtained $s = 31.4159$. The length of the chord AB is 31.4155 to six significant digits (found by using the law of sines, which we will cover in Chapter 7). As you can see, the first five digits in each number are the same. It seems reasonable then to approximate the length of chord AB with the length of arc AB.

As our next example illustrates, we can also use the procedure just outlined in the reverse order to find the radius of a circle by approximating arc length with the length of the associated chord.

EXAMPLE 4 A person standing on the earth notices that a 747 Jumbo Jet flying overhead subtends an angle of 0.45°. If the length of the jet is 230 feet, find its altitude to the nearest thousand feet.

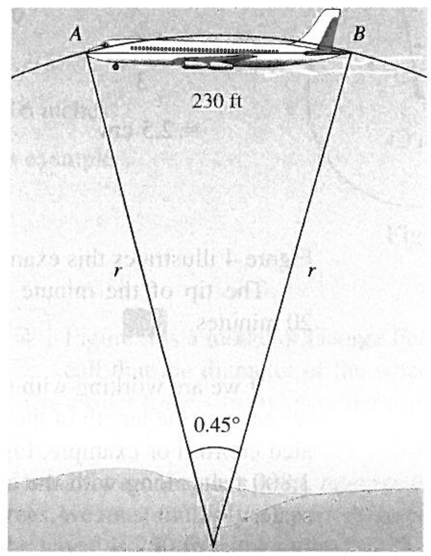


Figure 6

SOLUTION Figure 6 is a diagram of the situation. Since we are working with a relatively small angle in a circle with a large radius, we use the length of the airplane (chord AB in Figure 6) as an approximation of the length of the arc AB, and r as an approximation for the altitude of the plane.

Since $s = r\theta$, $r = \dfrac{s}{\theta}$

$$\text{so } r = \frac{230}{(0.45)(\pi/180)} \qquad \text{We multiply } 0.45° \text{ by } \pi/180 \text{ to change to radian measure}$$

$$= \frac{230(180)}{(0.45)(\pi)}$$

$$= 29{,}000 \text{ ft} \qquad \text{To the nearest thousand feet}$$

AREA OF A SECTOR

Next we want to derive the formula for the area of the sector formed by a central angle θ (Figure 7).

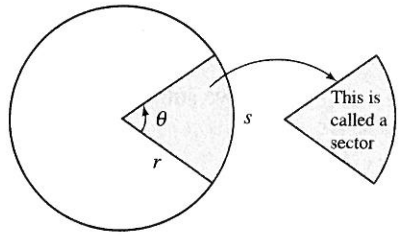


Figure 7

If we let A represent the area of the sector formed by central angle θ, we can find A by setting up a proportion as follows: We say the area A of the sector is to the area of the circle as θ is to one full rotation. That is,

$$\begin{array}{rcl}\text{Area of sector} \longrightarrow & \dfrac{A}{\pi r^2} = \dfrac{\theta}{2\pi} & \longleftarrow \text{Central angle } \theta \\ \text{Area of circle} \longrightarrow & & \longleftarrow \text{One full rotation}\end{array}$$

We solve for A by multiplying both sides of the proportion by πr^2.

$$\pi r^2 \cdot \frac{A}{\pi r^2} = \frac{\theta}{2\pi} \cdot \pi r^2$$

$$A = \frac{1}{2} r^2 \theta$$

AREA OF A SECTOR

If θ (in radians) is a central angle in a circle with radius r, then the area of the sector formed by angle θ is given by

$$A = \frac{1}{2} r^2 \theta \qquad (\theta \text{ in radians})$$

EXAMPLE 5 Find the area of the sector formed by a central angle of 1.4 radians in a circle of radius 2.1 meters.

SOLUTION We have $r = 2.1$ m and $\theta = 1.4$. Applying the formula for A gives us

$$\begin{aligned} A &= \frac{1}{2} r^2 \theta \\ &= \frac{1}{2}(2.1)^2(1.4) \\ &= 3.1 \text{ m}^2 \qquad \text{To the nearest tenth} \end{aligned}$$

REMEMBER Area is measured in square units. When $r = 2.1$ m, $r^2 = (2.1\text{ m})^2 = 4.41\text{ m}^2$.

EXAMPLE 6 If the sector formed by a central angle of 15° has an area of $\pi/3$ cm^2, find the radius of the circle.

SOLUTION We first convert 15° to radians.

$$\theta = 15\left(\frac{\pi}{180}\right) = \frac{\pi}{12}$$

Then we substitute $\theta = \pi/12$ and $A = \pi/3$ into the formula for A and solve for r.

$$A = \frac{1}{2}r^2\theta$$

$$\frac{\pi}{3} = \frac{1}{2}r^2\frac{\pi}{12}$$

$$\frac{\pi}{3} = \frac{\pi}{24}r^2$$

$$r^2 = \frac{\pi}{3} \cdot \frac{24}{\pi}$$

$$r^2 = 8$$

$$r = 2\sqrt{2} \text{ cm}$$

Note that we need only use the positive square root of 8, since we know our radius must be measured with positive units.

EXAMPLE 7 A lawn sprinkler located at the corner of a yard is set to rotate through 90° and project water out 30 feet. To three significant digits, what area of lawn is watered by the sprinkler?

SOLUTION We have $\theta = 90° = \frac{\pi}{2} \approx 1.57$ rad and $r = 30$ ft. Figure 8 illustrates this example.

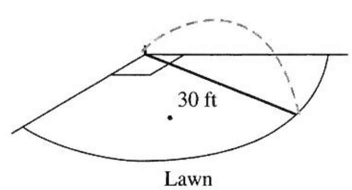


Figure 8

$$A = \frac{1}{2}r^2\theta$$

$$\approx \frac{1}{2}(30)^2(1.57)$$

$$= 707 \text{ ft}^2$$

GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

a. What is the definition and formula for arc length?

b. Which is longer, the chord AB or the corresponding arc AB?

c. What is a sector?

d. What is the definition and formula for the area of a sector?

PROBLEM SET 3.4

Unless otherwise stated, all answers in this Problem Set that need to be rounded should be rounded to three significant digits.

For each problem below, θ is a central angle in a circle of radius r. In each case, find the length of arc s cut off by θ.

1. $\theta = 2$, $r = 3$ inches

2. $\theta = 3$, $r = 2$ inches

3. $\theta = 1.5$, $r = 1.5$ ft

4. $\theta = 2.4$, $r = 1.8$ ft

5. $\theta = \pi/6$, $r = 12$ cm

6. $\theta = \pi/3$, $r = 12$ cm

7. $\theta = 60°$, $r = 4$ mm

8. $\theta = 30°$, $r = 4$ mm

9. $\theta = 240°$, $r = 10$ inches

10. $\theta = 315°$, $r = 5$ inches

11. Arc Length The minute hand of a clock is 2.4 centimeters long. How far does the tip of the minute hand travel in 20 minutes?

12. Arc Length The minute hand of a clock is 1.2 centimeters long. How far does the tip of the minute hand travel in 40 minutes?

13. Arc Length A space shuttle 200 miles above the earth is orbiting the earth once every 6 hours. How far does the shuttle travel in 1 hour? (Assume the radius of the earth is 4,000 miles.) Give your answer as both an exact value and an approximation to three significant digits (Figure 9).

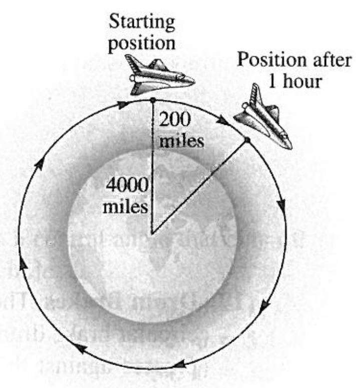
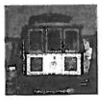


Figure 9

14. Arc Length How long, in hours, does it take the space shuttle in Problem 13 to travel 8,400 miles? Give both the exact value and an approximate value for your answer.

15. Arc Length The pendulum on a grandfather clock swings from side to side once every second. If the length of the pendulum is 4 feet and the angle through which it swings is 20°, how far does the tip of the pendulum travel in 1 second?

16. Arc Length Find the total distance traveled in 1 minute by the tip of the pendulum on the grandfather clock in Problem 15.



17. Cable Car Drive System The current San Francisco cable railway is driven by two large 14-foot-diameter drive wheels, called *sheaves*. Because of the figure-eight system used, the cable subtends a central angle of 270° on each sheave. Find the length of cable riding on one of the drive sheaves at any given time (Figure 10).

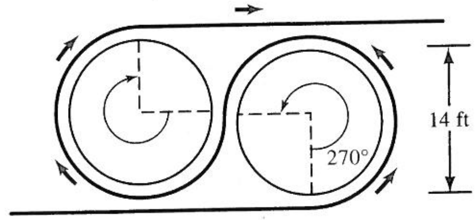


Figure 10

18. Cable Car Drive System The first cable railway to make use of the figure-eight drive system was the Sutter Street Railway in San Francisco in 1883 (Figure 11). Each drive sheave was 12 feet in diameter. Find the length of cable riding on one of the drive sheaves. (See Problem 17.)

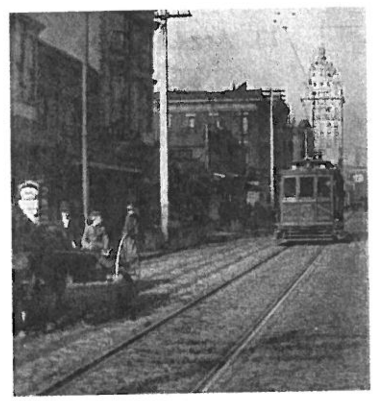


Figure 11

19. Drum Brakes The Isuzu NPR 250 light truck with manual transmission has a circular brake drum with a diameter of 320 millimeters. Each brake pad, which presses against the drum, is 307 millimeters long. What central angle is subtended by one of the brake pads? Write your answer in both radians and degrees (Figure 12).

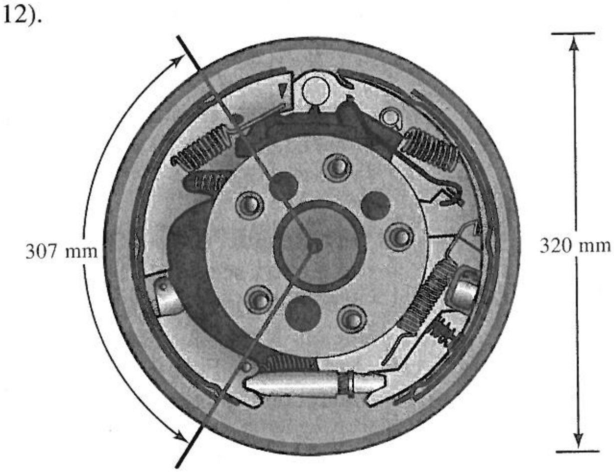


Figure 12

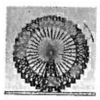
20. **Drum Brakes** The Isuzu NPR 250 truck with automatic transmission has a circular brake drum with a diameter of 320 millimeters. Each brake pad, which presses against the drum, is 335 millimeters long. What central angle is subtended by one of the brake pads? Write your answer in both radians and degrees.

21. **Diameter of the Moon** From the earth, the moon subtends an angle of approximately 0.5°. If the distance to the moon is approximately 240,000 miles, find an approximation for the diameter of the moon accurate to the nearest hundred miles. (See Example 4 and the discussion that precedes it.)

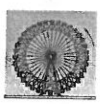
22. **Diameter of the Sun** If the distance to the sun is approximately 93 million miles, and, from the earth, the sun subtends an angle of approximately 0.5°, estimate the diameter of the sun to the nearest 10,000 miles.

Repeat Example 2 from this section for the following values of θ.

23. $\theta = 30°$
24. $\theta = 60°$
25. $\theta = 220°$
26. $\theta = 315°$



27. **Ferris Wheel** In Problem Set 2.3, we mentioned a Ferris wheel built in Vienna in 1897, known as the Great Wheel. The diameter of this wheel is 197 feet. Use Figure 3 from this section as a model of the Great Wheel, and find the distance traveled by a rider in going from initial position P_0 to position P_1 if
   a. θ is 60°
   b. θ is 210°
   c. θ is 285°



28. **Ferris Wheel** A Ferris Wheel called Colossus that we mentioned in Problem Set 2.3 has a diameter of 165 feet. Using Figure 3 from this section as a model, find the distance traveled by someone starting at initial position P_0 and moving to position P_1 if
   a. θ is 150°
   b. θ is 240°
   c. θ is 345°

In each problem below, θ is a central angle that cuts off an arc of length s. In each case, find the radius of the circle.

29. $\theta = 6$, $s = 3$ ft
30. $\theta = 1$, $s = 2$ ft
31. $\theta = 1.4$, $s = 4.2$ inches
32. $\theta = 5.1$, $s = 10.2$ inches
33. $\theta = \pi/4$, $s = \pi$ cm
34. $\theta = 3\pi/4$, $s = \pi$ cm
35. $\theta = 90°$, $s = \pi/2$ m
36. $\theta = 180°$, $s = \pi/2$ m
37. $\theta = 225°$, $s = 4$ km
38. $\theta = 150°$, $s = 5$ km

Find the area of the sector formed by central angle θ in a circle of radius r if

39. $\theta = 2$, $r = 3$ cm
40. $\theta = 3$, $r = 2$ cm
41. $\theta = 2.4$, $r = 4$ inches
42. $\theta = 1.8$, $r = 2$ inches
43. $\theta = \pi/5$, $r = 3$ m
44. $\theta = 2\pi/5$, $r = 3$ m
45. $\theta = 15°$, $r = 5$ m
46. $\theta = 15°$, $r = 10$ m

47. **Area of a Sector** A central angle of 2 radians cuts off an arc of length 4 inches. Find the area of the sector formed.

48. **Area of a Sector** An arc of length 3 feet is cut off by a central angle of $\pi/4$ radians. Find the area of the sector formed.

49. **Radius of a Circle** If the sector formed by a central angle of 30° has an area of $\pi/3$ square centimeters, find the radius of the circle.

50. Arc Length What is the length of the arc cut off by angle θ in Problem 49?

51. Radius of a Circle A sector of area $2\pi/3$ square inches is formed by a central angle of 45°. What is the radius of the circle?

52. Radius of a Circle A sector of area 25 square inches is formed by a central angle of 4 radians. Find the radius of the circle.

53. Lawn Sprinkler A lawn sprinkler is located at the corner of a yard. The sprinkler is set to rotate through 90° and project water out 60 feet. What is the area of the yard watered by the sprinkler?

54. Windshield Wiper An automobile windshield wiper 10 inches long rotates through an angle of 60°. If the rubber part of the blade covers only the last 9 inches of the wiper, find the area of the windshield cleaned by the windshield wiper.

55. Cycling The Shimano WH-R540 aluminum wheel, which has a diameter of 700 millimeters, has 8 pairs of spokes evenly distributed around the rim of the wheel. What is the length of the rim subtended by adjacent pairs of spokes (Figure 13)?

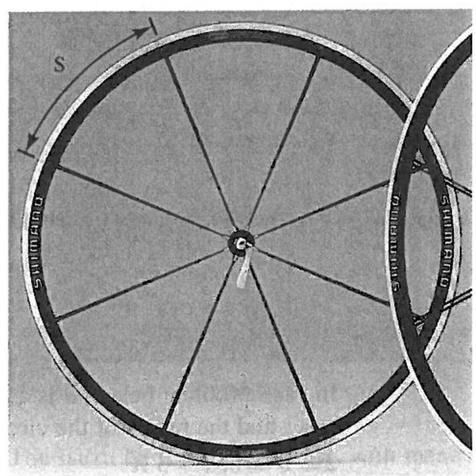


Figure 13

56. Cycling The Mavic Ksyrium Elite wheel, which has a diameter of 700 millimeters, has 18 spokes evenly distributed around the rim of the wheel. What is the length of the rim subtended by adjacent spokes (Figure 14)?

Figure 14

REVIEW PROBLEMS

The problems that follow review material we covered in Section 2.4.

57. Angle of Elevation If a 75-foot flagpole casts a shadow 43 feet long, what is the angle of elevation of the sun from the tip of the shadow?

58. Height of a Hill A road up a hill makes an angle of 5° with the horizontal. If the road from the bottom of the hill to the top of the hill is 2.5 miles long, how high is the hill?

59. Angle of Depression A person standing 5.2 feet from a mirror notices that the angle of depression from his eyes to the bottom of the mirror is 13°, while the angle of elevation to the top of the mirror is 12°. Find the vertical dimension of the mirror.

60. Distance and Bearing A boat travels on a course of bearing S 63°50′ E for 114 miles. How many miles south and how many miles east has the boat traveled?

61. Geometry The height of a right circular cone is 35.8 centimeters. If the diameter of the base is 20.5 centimeters, what angle does the side of the cone make with the base?

62. Distance and Bearing A ship leaves the harbor entrance and travels 35 miles in the direction N 42° E. The captain then turns the ship 90° and travels another 24 miles in the direction S 48° E. At that time, how far is the ship from the harbor entrance, and what is the bearing of the ship from the harbor entrance?

63. Angle of Depression A man standing on the roof of a building 86.0 feet above the ground looks down to the building next door. He finds the angle of depression to the roof of that building from the roof of his building to be 14.5°, while the angle of depression from the roof of his building to the bottom of the building next door is 43.2°. How tall is the building next door?

64. Height of a Tree Two people decide to find the height of a tree. They position themselves 35 feet apart in line with, and on the same side of, the tree. If they find the angles of elevation from the ground where they are standing to the top of the tree are 65° and 44°, how tall is the tree?

EXTENDING THE CONCEPTS

Apparent Diameter

As we mentioned in this section, for small central angles in circles with large radii, the intercepted arc and the chord are approximately the same length. Figure 15 shows a diagram of a person looking at the moon. The arc and the chord are essentially the same length and are both good approximations to the diameter.

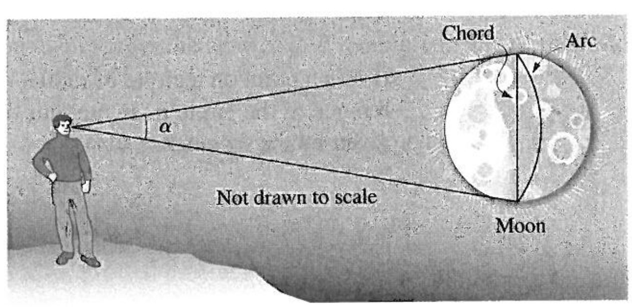


Figure 15

PROBLEM SET 3.4

1. 6 inches **3.** 2.25 ft **5.** 2π cm $\approx$ 6.28 cm **7.** $\frac{4\pi}{3}$ mm $\approx$ 4.19 mm **9.** $\frac{40\pi}{3}$ inches $\approx$ 41.9 inches

11. 5.03 cm **13.** 4,400 mi **15.** $\frac{4\pi}{9}$ ft $\approx$ 1.40 ft **17.** 33.0 feet **19.** 1.92 radians, 110° **21.** 2,100 mi

23. 65.4 ft **25.** 480 ft **27. a.** 103 ft **b.** 361 ft **c.** 490 ft **29.** 0.5 ft **31.** 3 inches **33.** 4 cm

35. 1 m **37.** $\frac{16}{5\pi}$ km $\approx$ 1.02 km **39.** 9 cm^2 **41.** 19.2 $inches^2$ **43.** $\frac{9\pi}{10}$ $m^2 \approx$ 2.83 m^2

45. $\frac{25\pi}{24}$ $m^2 \approx$ 3.27 m^2 **47.** 4 $inches^2$ **49.** 2 cm **51.** $\frac{4}{\sqrt{3}}$ inches $\approx$ 2.31 inches **53.** 900π $ft^2 \approx$ 2,830 ft^2

55. $\frac{175\pi}{2}$ mm $\approx$ 275 mm **57.** 60.2° **59.** 2.31 ft **61.** 74.0° **63.** 62.3 ft **65.** 0.009 radian $\approx$ 0.518°

67. The sun is also about 400 times farther away from the earth.

PROBLEM SET 3.5

1. 1.5 ft/min **3.** 3 cm/sec **5.** 15 mi/hr **7.** 80 ft **9.** 22.5 mi **11.** 7 mi **13.** $\frac{2\pi}{15}$ rad/sec $\approx$ 0.419 rad/sec

15. 4 rad/min **17.** $\frac{8}{3}$ rad/sec $\approx$ 2.67 rad/sec **19.** 37.5π rad/hr $\approx$ 118 rad/hr

21. $d = 100 \tan \frac{\pi}{2}t$; when $t = \frac{1}{2}$, $d = 100$ ft; when $t = \frac{3}{2}$, $d = -100$ ft; when $t = 1$, d is undefined because the light rays are parallel to the wall. **23.** 40 inches **25.** 180π m $\approx$ 565 m **27.** 4,500 ft **29.** 20π rad/min $\approx$ 62.8 rad/min

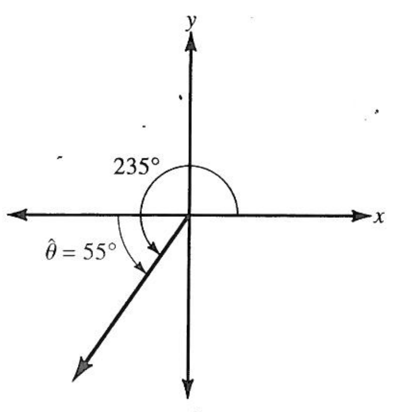
31. $\frac{200\pi}{3}$ rad/min $\approx$ 209 rad/min **33.** 11.6π rad/min $\approx$ 36.4 rad/min **35.** 10 inches/sec **37.** 0.5 rad/sec

39. 80π ft/min $\approx$ 251 ft/min **41.** $\frac{\pi}{12}$ rad/hr $\approx$ 0.262 rad/hr **43.** 300π ft/min $\approx$ 942 ft/min **45.** 9.50 mi/hr

47. 23.3 rpm **49.** 5.65 ft/sec **51.** 0.47 mi/hr **53.** 889 rad/min (53,300 rad/hr) **55.** 18.8 km/hr
57. 52.3 mm **59.** 80.8 rpm **61.** 10.4 mi/hr at N 24.1° W **63.** $|\mathbf{V}_x| = 54.3$ ft/sec, $|\mathbf{V}_y| = 40.9$ ft/sec
65. 46.2 mi south, 71.9 mi west **67.** See the Solutions Manual.

CHAPTER 3 TEST

1.



2.