SECTION 5.3 DOUBLE-ANGLE FORMULAS

We will begin this section by deriving the formulas for $\sin 2A$ and $\cos 2A$ using the formulas for $\sin(A + B)$ and $\cos(A + B)$. The formulas we derive for $\sin 2A$ and $\cos 2A$ are called *double-angle* formulas. Here is the derivation of the formula for $\sin 2A$.

$\sin 2A$	$= \sin(A + A)$	Write $2A$ as $A + A$
	$= \sin A \cos A + \cos A \sin A$	Sum formula
	$= \sin A \cos A + \sin A \cos A$	Commutative property
	$= 2 \sin A \cos A$	

The last line gives us our first double-angle formula.

$$\boxed{\sin 2A = 2 \sin A \cos A}$$

The first thing to notice about this formula is that it indicates the 2 in $\sin 2A$ *cannot* be factored out and written as a coefficient. That is,

$$\sin 2A \neq 2 \sin A$$

For example, if $A = 30°$, $\sin 2 \cdot 30° = \sin 60° = \sqrt{3}/2$, which is not the same as $2 \sin 30° = 2\left(\frac{1}{2}\right) = 1$.

EXAMPLE 1 If $\sin A = \frac{3}{5}$ with A in QII, find $\sin 2A$.

SOLUTION In order to apply the formula for $\sin 2A$, we must first find $\cos A$. Since A terminates in QII, $\cos A$ is negative.

$$\cos A = -\sqrt{1 - \sin^2 A} = -\sqrt{1 - \left(\frac{3}{5}\right)^2} = -\sqrt{\frac{16}{25}} = -\frac{4}{5}$$

Now we can apply the formula for $\sin 2A$.

$$\begin{aligned} \sin 2A &= 2 \sin A \cos A \\ &= 2\left(\frac{3}{5}\right)\left(-\frac{4}{5}\right) \\ &= -\frac{24}{25} \end{aligned}$$

We can also use our new formula to expand the work we did previously with identities.

EXAMPLE 2 Prove $(\sin \theta + \cos \theta)^2 = 1 + \sin 2\theta$.

PROOF

$(\sin \theta + \cos \theta)^2$	$= \sin^2 \theta + 2 \sin \theta \cos \theta + \cos^2 \theta$	Expand
	$= 1 + 2 \sin \theta \cos \theta$	Pythagorean identity
	$= 1 + \sin 2\theta$	Double-angle identity

EXAMPLE 3 Prove $\sin 2x = \dfrac{2\cot x}{1 + \cot^2 x}$.

PROOF

$$\frac{2\cot x}{1+\cot^2 x} = \frac{2 \cdot \dfrac{\cos x}{\sin x}}{1 + \dfrac{\cos^2 x}{\sin^2 x}} \qquad \text{Ratio identity}$$

$$= \frac{2\sin x\cos x}{\sin^2 x + \cos^2 x} \qquad \text{Multiply numerator and denominator by } \sin^2 x$$

$$= 2\sin x\cos x \qquad \text{Pythagorean identity}$$

$$= \sin 2x \qquad \text{Double-angle identity}$$

There are three forms of the double-angle formula for $\cos 2A$. The first involves both $\sin A$ and $\cos A$, the second involves only $\cos A$, and the third involves only $\sin A$. Here is how we obtain the three formulas.

$$\cos 2A = \cos(A + A) \qquad \text{Write } 2A \text{ as } A + A$$

$$= \cos A\cos A - \sin A\sin A \qquad \text{Sum formula}$$

$$= \cos^2 A - \sin^2 A$$

To write this last formula in terms of $\cos A$ only, we substitute $1 - \cos^2 A$ for $\sin^2 A$.

$$\cos 2A = \cos^2 A - (1 - \cos^2 A)$$

$$= \cos^2 A - 1 + \cos^2 A$$

$$= 2\cos^2 A - 1$$

To write the formula in terms of $\sin A$ only, we substitute $1 - \sin^2 A$ for $\cos^2 A$ in the last line above.

$$\cos 2A = 2\cos^2 A - 1$$

$$= 2(1 - \sin^2 A) - 1$$

$$= 2 - 2\sin^2 A - 1$$

$$= 1 - 2\sin^2 A$$

Here are the three forms of the double-angle formula for $\cos 2A$.

$\cos 2A = \cos^2 A - \sin^2 A$	First form
$= 2\cos^2 A - 1$	Second form
$= 1 - 2\sin^2 A$	Third form

Which form we choose will depend on the circumstances of the problem, as the next three examples illustrate.

EXAMPLE 4 If $\sin A = 1/\sqrt{5}$, find $\cos 2A$.

SOLUTION In this case, since we are given $\sin A$, applying the third form of the formula for $\cos 2A$ will give us the answer more quickly than applying either of the other two forms.

$$\begin{aligned}\cos 2A &= 1 - 2\sin^2 A \\ &= 1 - 2\left(\frac{1}{\sqrt{5}}\right)^2 \\ &= 1 - \frac{2}{5} \\ &= \frac{3}{5}\end{aligned}$$

EXAMPLE 5 Prove $\cos 4x = 8\cos^4 x - 8\cos^2 x + 1$.

PROOF We can write $\cos 4x$ as $\cos(2 \cdot 2x)$ and apply our double-angle formula. Since the right side is written in terms of $\cos x$ only, we will choose the second form of our double-angle formula for $\cos 2A$.

$$\begin{aligned}\cos 4x &= \cos(2 \cdot 2x) && \\ &= 2\cos^2 2x - 1 && \text{Double-angle formula} \\ &= 2(2\cos^2 x - 1)^2 - 1 && \text{Double-angle formula} \\ &= 2(4\cos^4 x - 4\cos^2 x + 1) - 1 && \text{Square} \\ &= 8\cos^4 x - 8\cos^2 x + 2 - 1 && \text{Distribute} \\ &= 8\cos^4 x - 8\cos^2 x + 1 && \text{Simplify}\end{aligned}$$

EXAMPLE 6 Graph $y = 3 - 6\sin^2 x$ from $x = 0$ to $x = 2\pi$.

SOLUTION To write the equation in the form $y = A\cos Bx$, we factor 3 from each term on the right side and then apply the formula for $\cos 2A$ to the remaining expression to write it as a single trigonometric function.

$$\begin{aligned}y &= 3 - 6\sin^2 x && \\ &= 3(1 - 2\sin^2 x) && \text{Factor 3 from each term} \\ &= 3\cos 2x && \text{Double-angle formula}\end{aligned}$$

The graph of $y = 3\cos 2x$ will have an amplitude of 3 and a period of $2\pi/2 = \pi$. The graph is shown in Figure 1.

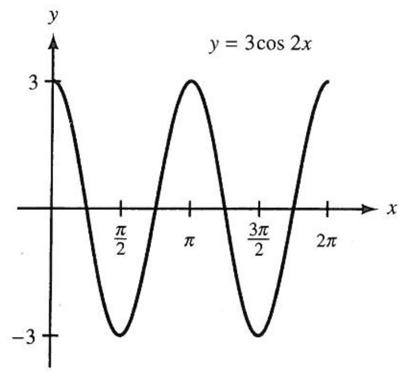


Figure 1

EXAMPLE 7 Prove $\tan\theta = \dfrac{1-\cos 2\theta}{\sin 2\theta}$.

PROOF

$$\frac{1-\cos 2\theta}{\sin 2\theta} = \frac{1-(1-2\sin^2\theta)}{2\sin\theta\cos\theta} \qquad \text{Double-angle formulas}$$

$$= \frac{2\sin^2\theta}{2\sin\theta\cos\theta} \qquad \text{Simplify numerator}$$

$$= \frac{\sin\theta}{\cos\theta} \qquad \text{Divide out common factor } 2\sin\theta$$

$$= \tan\theta \qquad \text{Ratio identity}$$

We end this section by deriving the formula for tan 2A.

$$\tan 2A = \tan(A + A)$$

$$= \frac{\tan A + \tan A}{1 - \tan A\tan A}$$

$$= \frac{2\tan A}{1-\tan^2 A}$$

Our double-angle formula for tan 2A is

$$\boxed{\tan 2A = \frac{2\tan A}{1-\tan^2 A}}$$

EXAMPLE 8 Simplify $\dfrac{2\tan 15°}{1-\tan^2 15°}$.

SOLUTION The expression has the same form as the right side of our double-angle formula for tan 2A. Therefore,

$$\frac{2\tan 15°}{1-\tan^2 15°} = \tan(2\cdot 15°)$$

$$= \tan 30°$$

$$= \frac{1}{\sqrt{3}}$$

EXAMPLE 9 If $x = 3 \tan \theta$, write the expression below in terms of just x.

$$\frac{\theta}{2} + \frac{\sin 2\theta}{4}$$

SOLUTION To substitute for the first term above, we need to write θ in terms of x. To do so, we solve the equation $x = 3 \tan \theta$ for θ.

$$\text{If} \qquad 3 \tan \theta = x$$

$$\text{then} \qquad \tan \theta = \frac{x}{3}$$

$$\text{and} \qquad \theta = \tan^{-1} \frac{x}{3}$$

Next, since the inverse tangent function can take on values only between $-\pi/2$ and $\pi/2$, we can visualize θ by drawing a right triangle in which θ is one of the acute angles. Since $\tan \theta = x/3$, we label the side opposite θ with x and the side adjacent to θ with 3.

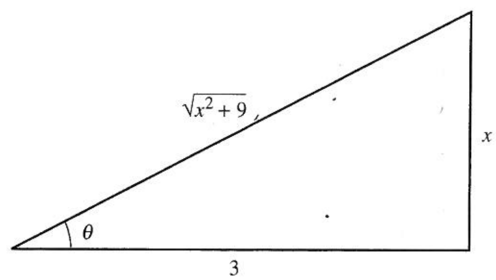


Figure 2

By the Pythagorean Theorem, the hypotenuse of the triangle in Figure 2 must be $\sqrt{x^2 + 9}$, which means $\sin \theta = x/\sqrt{x^2 + 9}$ and $\cos \theta = 3/\sqrt{x^2 + 9}$. Now we are ready to simplify and substitute to solve our problem.

$$\begin{aligned}
\frac{\theta}{2} + \frac{\sin 2\theta}{4} &= \frac{\theta}{2} + \frac{2 \sin \theta \cos \theta}{4} \\
&= \frac{\theta}{2} + \frac{\sin \theta \cos \theta}{2} \\
&= \frac{1}{2}(\theta + \sin \theta \cos \theta) \\
&= \frac{1}{2}\left(\tan^{-1} \frac{x}{3} + \frac{x}{\sqrt{x^2 + 9}} \cdot \frac{3}{\sqrt{x^2 + 9}}\right) \\
&= \frac{1}{2}\left(\tan^{-1} \frac{x}{3} + \frac{3x}{x^2 + 9}\right)
\end{aligned}$$

Note that we do not have to use absolute value symbols when we multiply and simplify the square roots in the second to the last line above because we know that $x^2 + 9$ is always positive.

GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

a. What is the formula for $\sin 2A$?

b. What are the three formulas for $\cos 2A$?

c. What is the formula for $\tan 2A$?

d. As a general rule, when do we use θ to indicate an angle, and when do we use A, B, or other non-Greek symbols? (*Hint:* Think in terms of degrees and radians.)

PROBLEM SET 5.3

Let $\sin A = -\frac{3}{5}$ with A in QIII and find

1. $\sin 2A$

2. $\cos 2A$

3. $\tan 2A$

4. $\cot 2A$

Let $\cos x = 1/\sqrt{10}$ with x in QIV and find

5. $\cos 2x$

6. $\sin 2x$

7. $\cot 2x$

8. $\tan 2x$

Let $\tan \theta = \frac{5}{12}$ with θ in QI and find

9. $\sin 2\theta$

10. $\cos 2\theta$

11. $\csc 2\theta$

12. $\sec 2\theta$

Let $\csc t = \sqrt{5}$ with t in QII and find

13. $\cos 2t$

14. $\sin 2t$

15. $\sec 2t$

16. $\csc 2t$

Graph each of the following from $x = 0$ to $x = 2\pi$.

17. $y = 4 - 8 \sin^2 x$

18. $y = 2 - 4 \sin^2 x$

19. $y = 6 \cos^2 x - 3$

20. $y = 4 \cos^2 x - 2$

21. $y = 1 - 2 \sin^2 2x$

22. $y = 2 \cos^2 2x - 1$

Use exact values to show that each of the following is true.

23. $\sin 60° = 2 \sin 30° \cos 30°$

24. $\cos 60° = 1 - 2 \sin^2 30°$

25. $\cos 120° = \cos^2 60° - \sin^2 60°$

26. $\sin 90° = 2 \sin 45° \cos 45°$

27. If $\tan A = \frac{3}{4}$, find $\tan 2A$.

28. If $\tan A = -\sqrt{3}$, find $\tan 2A$.

Simplify each of the following.

29. $2 \sin 15° \cos 15°$

30. $\cos^2 15° - \sin^2 15°$

31. $1 - 2 \sin^2 75°$

32. $2 \cos^2 105° - 1$

33. $\sin \dfrac{\pi}{12} \cos \dfrac{\pi}{12}$

34. $\sin \dfrac{\pi}{8} \cos \dfrac{\pi}{8}$

35. $\dfrac{\tan 22.5°}{1 - \tan^2 22.5°}$

36. $\dfrac{\tan \frac{3\pi}{8}}{1 - \tan^2 \frac{3\pi}{8}}$

Prove each of the following identities.

37. $(\sin x - \cos x)^2 = 1 - \sin 2x$

38. $(\cos x - \sin x)(\cos x + \sin x) = \cos 2x$

39. $\cos^2 \theta = \dfrac{1 + \cos 2\theta}{2}$

40. $\sin^2 \theta = \dfrac{1 - \cos 2\theta}{2}$

41. $\cot \theta = \dfrac{\sin 2\theta}{1 - \cos 2\theta}$

42. $\cos 2\theta = \dfrac{1 - \tan^2 \theta}{1 + \tan^2 \theta}$

43. $2 \csc 2x = \tan x + \cot x$

44. $2 \cot 2x = \cot x - \tan x$

45. $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$

46. $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$

47. $\cos^4 x - \sin^4 x = \cos 2x$

48. $2 \sin^4 x + 2 \sin^2 x \cos^2 x = 1 - \cos 2x$

49. $\cot \theta - \tan \theta = \dfrac{\cos 2\theta}{\sin \theta \cos \theta}$

50. $\csc \theta - 2 \sin \theta = \dfrac{\cos 2\theta}{\sin \theta}$

51. $\sin 4A = 4 \sin A \cos^3 A - 4 \sin^3 A \cos A$

52. $\cos 4A = \cos^4 A - 6 \cos^2 A \sin^2 A + \sin^4 A$

53. $\dfrac{1 - \tan x}{1 + \tan x} = \dfrac{1 - \sin 2x}{\cos 2x}$

54. $\dfrac{2 - 2 \cos 2x}{\sin 2x} = \sec x \csc x - \cot x + \tan x$

Use your graphing calculator to determine if each equation appears to be an identity by graphing the left expression and right expression together. If so, then prove the identity.

55. $\cot 2x = \dfrac{\cos x - \sin x \tan x}{\sec x \sin 2x}$

56. $\sec 2x = \dfrac{\sec^2 x \csc^2 x}{\csc^2 x - \sec^2 x}$

57. $\sec 2x = \dfrac{\sec^2 x \csc^2 x}{\csc^2 x + \sec^2 x}$

58. $\csc 2x = \dfrac{\sec x + \csc x}{2 \sin x - 2 \cos x}$

59. $\csc 2x = \dfrac{\sec x + \csc x}{2 \sin x + 2 \cos x}$

60. $\tan 2x = \dfrac{2 \cot x}{\csc^2 x - 2}$

61. If $x = 5 \tan \theta$, write the expression $\dfrac{\theta}{2} - \dfrac{\sin 2\theta}{4}$ in terms of just x.

62. If $x = 4 \sin \theta$, write the expression $\dfrac{\theta}{2} - \dfrac{\sin 2\theta}{4}$ in terms of just x.

63. If $x = 3 \sin \theta$, write the expression $\dfrac{\theta}{2} - \dfrac{\sin 2\theta}{4}$ in terms of just x.

64. If $x = 2 \sin \theta$, write the expression $2\theta - \tan 2\theta$ in terms of just x.

REVIEW PROBLEMS

The problems that follow review material we covered in Section 4.5. Graph each of the following from $x = 0$ to $x = 4\pi$.

65. $y = 2 - 2 \cos x$

66. $y = 2 + 2 \cos x$

67. $y = 3 - 3 \cos x$

68. $y = 3 + 3 \cos x$

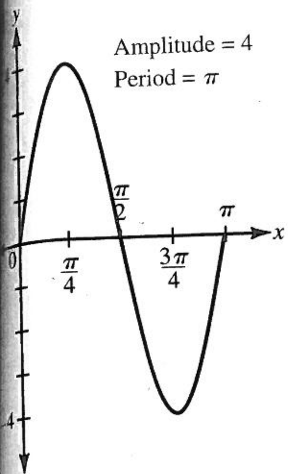
69. $y = \cos x + \dfrac{1}{2} \sin 2x$

70. $y = \sin x + \dfrac{1}{2} \cos 2x$

Graph each of the following from $x = 0$ to $x = 8$.

71. $y = \dfrac{1}{2}x + \sin \pi x$

72. $y = x + \sin \dfrac{\pi}{2}x$



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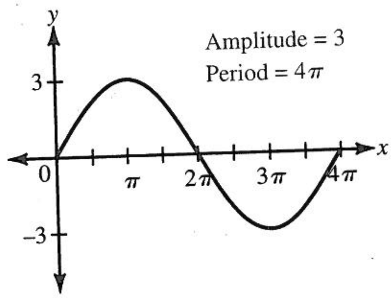

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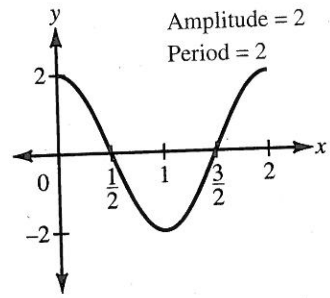

1. See the solution to Problem 9 in Problem Set 4.2.

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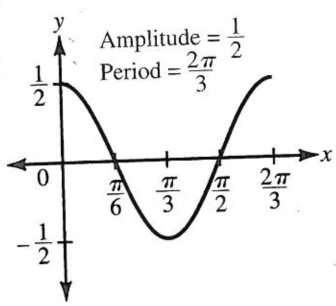

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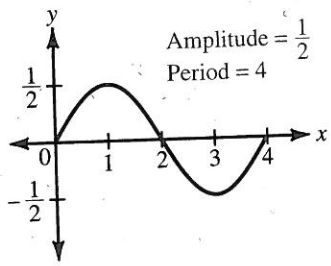
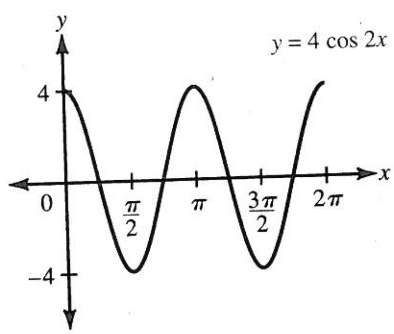


PROBLEM SET 5.3

1. $\frac{24}{25}$ **3.** $\frac{24}{7}$ **5.** $-\frac{4}{5}$ **7.** $\frac{4}{3}$ **9.** $\frac{120}{169}$ **11.** $\frac{169}{120}$ **13.** $\frac{3}{5}$ **15.** $\frac{5}{3}$

17. $y = 4 - 8\sin^2 x$
$= 4(1 - 2\sin^2 x)$
$= 4\cos 2x$



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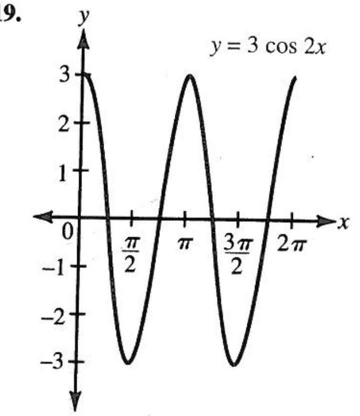

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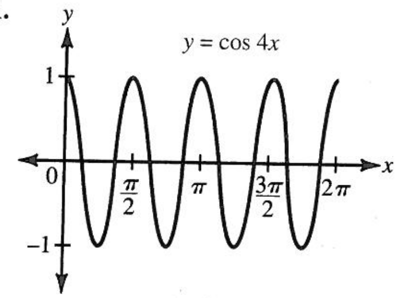


For Problems 23 and 25, see the Solutions Manual.

27. $\frac{24}{7}$ **29.** $\frac{1}{2}$ **31.** $-\frac{\sqrt{3}}{2}$ **33.** $\frac{1}{4}$ **35.** $\frac{1}{2}$

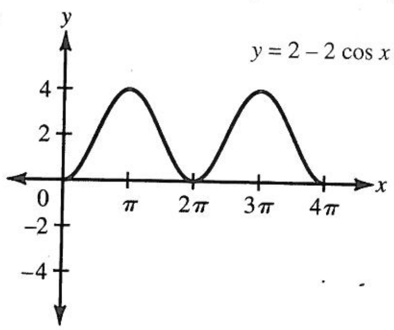
For Problems 37–53, solutions are given in the Solutions Manual.

NOTE For Problems 55–59, when the equation is an identity, the proof is given in the Solutions Manual.

55. Is an identity **57.** Not an identity **59.** Is an identity

61. $\frac{1}{2}\left(\tan^{-1}\frac{x}{5} - \frac{5x}{x^2+25}\right)$ **63.** $\frac{1}{2}\left(\sin^{-1}\frac{x}{3} - \frac{x\sqrt{9-x^2}}{9}\right)$

65.



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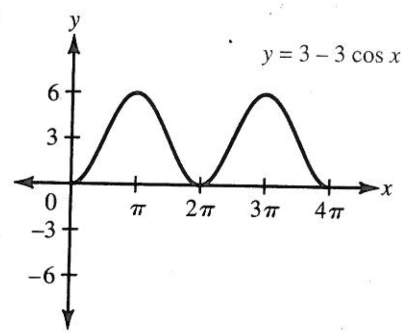

69. See the solution to Problem 21 in Problem Set 4.5.

71.