SECTION 5.4 HALF-ANGLE FORMULAS

In this section, we will derive formulas for $\sin \frac{A}{2}$ and $\cos \frac{A}{2}$. These formulas are called *half-angle* formulas and are derived from the double-angle formulas for $\cos 2A$.

In Section 5.3, we developed three ways to write the formula for $\cos 2A$, two of which were

$$\cos 2A = 1 - 2\sin^2 A \qquad \text{and} \qquad \cos 2A = 2\cos^2 A - 1$$

Since the choice of the letter we use to denote the angles in these formulas is arbitrary, we can use an x instead of A.

$$\cos 2x = 1 - 2\sin^2 x \qquad \text{and} \qquad \cos 2x = 2\cos^2 x - 1$$

Let us exchange sides in the first formula and solve for $\sin x$.

$1 - 2\sin^2 x = \cos 2x$	Exchange sides
$-2\sin^2 x = -1 + \cos 2x$	Add -1 to both sides
$\sin^2 x = \dfrac{1 - \cos 2x}{2}$	Divide both sides by -2
$\sin x = \pm\sqrt{\dfrac{1 - \cos 2x}{2}}$	Take the square root of both sides

Since every value of x can be written as $\frac{1}{2}$ of some other number A, we can replace x with $A/2$. This is equivalent to saying $2x = A$.

$$\boxed{\sin \frac{A}{2} = \pm\sqrt{\frac{1 - \cos A}{2}}}$$

This last expression is the half-angle formula for $\sin \frac{A}{2}$. To find the half-angle formula for $\cos \frac{A}{2}$, we solve $\cos 2x = 2\cos^2 x - 1$ for $\cos x$ and then replace x with $A/2$ (and $2x$ with A). Without showing the steps involved in this process, here is the result:

$$\boxed{\cos \frac{A}{2} = \pm\sqrt{\frac{1 + \cos A}{2}}}$$

In both half-angle formulas, the sign in front of the radical, + or −, is determined by the quadrant in which $A/2$ terminates.

EXAMPLE 1 If $\cos A = \frac{3}{5}$ with $270° < A < 360°$, find $\sin \frac{A}{2}$, $\cos \frac{A}{2}$, and $\tan \frac{A}{2}$.

SOLUTION First of all, we determine the quadrant in which $A/2$ terminates.

$$270° < A < 360° \Rightarrow \frac{270°}{2} < \frac{A}{2} < \frac{360°}{2}$$

$$\text{or} \qquad 135° < \frac{A}{2} < 180° \Rightarrow \frac{A}{2} \in \text{QII}$$

In quadrant II, sine is positive, and cosine and tangent are negative.

$$\sin\frac{A}{2} = \sqrt{\frac{1-\cos A}{2}} \qquad \cos\frac{A}{2} = -\sqrt{\frac{1+\cos A}{2}}$$

$$= \sqrt{\frac{1-3/5}{2}} \qquad = -\sqrt{\frac{1+3/5}{2}}$$

$$= \sqrt{\frac{1}{5}} \qquad = -\sqrt{\frac{4}{5}}$$

$$= \frac{1}{\sqrt{5}} \qquad = -\frac{2}{\sqrt{5}}$$

$$\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \frac{\frac{1}{\sqrt{5}}}{-\frac{2}{\sqrt{5}}} = -\frac{1}{2}$$

EXAMPLE 2 If $\sin A = -\frac{12}{13}$ with $180° < A < 270°$, find the six trigonometric functions of $A/2$.

SOLUTION To use the half-angle formulas, we need to find $\cos A$.

Because $A \in$ QIII
↓

$$\cos A = -\sqrt{1-\sin^2 A} = -\sqrt{1-\left(-\frac{12}{13}\right)^2} = -\sqrt{\frac{25}{169}} = -\frac{5}{13}$$

Also, $A/2$ terminates in QII because

$$180° < A < 270°$$

$$\frac{180°}{2} < \frac{A}{2} < \frac{270°}{2}$$

$$90° < \frac{A}{2} < 135° \Rightarrow \frac{A}{2} \in \text{QII}$$

In quadrant II, sine is positive and cosine is negative.

$$\sin\frac{A}{2} = \sqrt{\frac{1-(-5/13)}{2}} \qquad \cos\frac{A}{2} = -\sqrt{\frac{1+(-5/13)}{2}}$$

$$= \sqrt{\frac{9}{13}} \qquad = -\sqrt{\frac{4}{13}}$$

$$= \frac{3}{\sqrt{13}} \qquad = -\frac{2}{\sqrt{13}}$$

Now that we have sine and cosine of $A/2$, we can apply the ratio identity for tangent to find $\tan\frac{A}{2}$.

$$\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \frac{\frac{3}{\sqrt{13}}}{-\frac{2}{\sqrt{13}}} = -\frac{3}{2}$$

Next, we apply our reciprocal identities to find cosecant, secant, and cotangent of $A/2$.

$$\csc\frac{A}{2}=\frac{1}{\sin\frac{A}{2}}=\frac{\sqrt{13}}{3}\qquad \sec\frac{A}{2}=\frac{1}{\cos\frac{A}{2}}=-\frac{\sqrt{13}}{2}$$

$$\cot\frac{A}{2}=\frac{1}{\tan\frac{A}{2}}=-\frac{2}{3}$$

In the previous two examples, we found $\tan\frac{A}{2}$ by using the ratio of $\sin\frac{A}{2}$ to $\cos\frac{A}{2}$. There are formulas that allow us to find $\tan\frac{A}{2}$ directly from $\sin A$ and $\cos A$. In Example 7 of Section 5.3, we proved the following identity:

$$\tan\theta=\frac{1-\cos 2\theta}{\sin 2\theta}$$

If we let $\theta=\frac{A}{2}$ in this identity, we obtain a formula for $\tan\frac{A}{2}$ that involves only $\sin A$ and $\cos A$. Here it is.

$$\tan\frac{A}{2}=\frac{1-\cos A}{\sin A}$$

If we multiply the numerator and denominator of the right side of this formula by $1+\cos A$ and simplify the result, we have a second formula for $\tan\frac{A}{2}$.

$$\tan\frac{A}{2}=\frac{\sin A}{1+\cos A}$$

EXAMPLE 3 Find tan 15°.

SOLUTION Since $15°=30°/2$, we can use a half-angle formula to find tan 15°.

$$\begin{aligned}\tan 15° &= \tan\frac{30°}{2}\\ &=\frac{1-\cos 30°}{\sin 30°}\\ &=\frac{1-\frac{\sqrt{3}}{2}}{\frac{1}{2}}\\ &=2-\sqrt{3}\end{aligned}$$

EXAMPLE 4 Graph $y = 4\cos^2\frac{x}{2}$ from $x = 0$ to $x = 4\pi$.

SOLUTION Applying our half-angle formula for $\cos\frac{x}{2}$ to the right side, we have

$$\begin{aligned} y &= 4\cos^2\frac{x}{2} \\ &= 4\left(\pm\sqrt{\frac{1+\cos x}{2}}\right)^2 \\ &= 4\left(\frac{1+\cos x}{2}\right) \\ &= 2 + 2\cos x \end{aligned}$$

We graph $y = 2 + 2\cos x$ using the method developed in Section 4.5. The graph is shown in Figure 1.

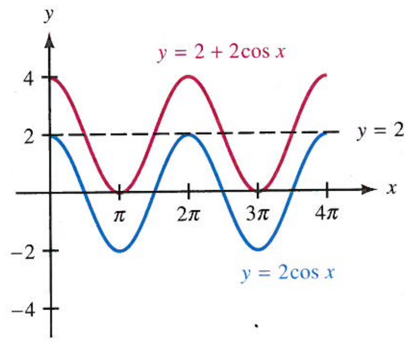


Figure 1

EXAMPLE 5 Prove $\sin^2\frac{x}{2} = \frac{\tan x - \sin x}{2\tan x}$.

PROOF We can use a half-angle formula on the left side. In this case, since we have $\sin^2(x/2)$, we write the half-angle formula without the square root sign. After that, we multiply the numerator and denominator on the left side by $\tan x$ because the right side has $\tan x$ in both the numerator and the denominator.

$\sin^2\frac{x}{2} = \frac{1-\cos x}{2}$	Square of half-angle formula
$= \frac{\mathbf{tan}\,\boldsymbol{x}}{\mathbf{tan}\,\boldsymbol{x}} \cdot \frac{1-\cos x}{2}$	Multiply numerator and denominator by $\tan x$
$= \frac{\tan x - \tan x\cos x}{2\tan x}$	Distributive property
$= \frac{\tan x - \sin x}{2\tan x}$	$\tan x\cos x$ is $\sin x$

GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

a. From what other formulas are half-angle formulas derived?

b. What is the formula for $\sin \frac{A}{2}$?

c. What is the formula for $\cos \frac{A}{2}$?

d. What are the two formulas for $\tan \frac{A}{2}$?

PROBLEM SET 5.4

NOTE For the following problems, assume that all the given angles are in simplest form, so that if A is in QIV you may assume that $270° < A < 360°$.

If $\cos A = \frac{1}{2}$ with A in QIV, find

1. $\sin \frac{A}{2}$

2. $\cos \frac{A}{2}$

3. $\csc \frac{A}{2}$

4. $\sec \frac{A}{2}$

If $\sin A = -\frac{3}{5}$ with A in QIII, find

5. $\cos \frac{A}{2}$

6. $\sin \frac{A}{2}$

7. $\sec \frac{A}{2}$

8. $\csc \frac{A}{2}$

If $\sin B = -\frac{1}{3}$ with B in QIII, find

9. $\sin \frac{B}{2}$

10. $\csc \frac{B}{2}$

11. $\cos \frac{B}{2}$

12. $\sec \frac{B}{2}$

13. $\tan \frac{B}{2}$

14. $\cot \frac{B}{2}$

If $\sin A = \frac{4}{5}$ with A in QII, and $\sin B = \frac{3}{5}$ with B in QI, find

15. $\sin \frac{A}{2}$

16. $\cos \frac{A}{2}$

17. $\cos 2A$

18. $\sin 2A$

19. $\sec 2A$

20. $\csc 2A$

21. $\cos \frac{B}{2}$

22. $\sin \frac{B}{2}$

23. $\sin (A + B)$

24. $\cos (A + B)$

25. $\cos (A - B)$

26. $\sin (A - B)$

Graph each of the following from $x = 0$ to $x = 4\pi$.

27. $y = 4 \sin^2 \frac{x}{2}$

28. $y = 6 \cos^2 \frac{x}{2}$

29. $y = 2 \cos^2 \frac{x}{2}$

30. $y = 2 \sin^2 \frac{x}{2}$

Use half-angle formulas to find exact values for each of the following:

31. $\cos 15°$

32. $\tan 15°$

33. $\sin 75°$

34. $\cos 75°$

35. $\cos 105°$

36. $\sin 105°$

Prove the following identities.

37. $\sin^2 \frac{\theta}{2} = \frac{\csc \theta - \cot \theta}{2 \csc \theta}$

38. $2 \cos^2 \frac{\theta}{2} = \frac{\sin^2 \theta}{1 - \cos \theta}$

39. $\sec^2 \frac{A}{2} = \frac{2 \sec A}{\sec A + 1}$

40. $\csc^2 \frac{A}{2} = \frac{2 \sec A}{\sec A - 1}$

41. $\tan \frac{B}{2} = \csc B - \cot B$

42. $\tan \frac{B}{2} = \frac{\sec B}{\sec B \csc B + \csc B}$

43. $\tan \frac{x}{2} + \cot \frac{x}{2} = 2 \csc x$

44. $\tan \frac{x}{2} - \cot \frac{x}{2} = -2 \cot x$

45. $\cos^2 \frac{\theta}{2} = \frac{\tan \theta + \sin \theta}{2 \tan \theta}$

46. $2 \sin^2 \frac{\theta}{2} = \frac{\sin^2 \theta}{1 + \cos \theta}$

47. $\cos^4 \theta = \frac{1}{4} + \frac{\cos 2\theta}{2} + \frac{\cos^2 2\theta}{4}$

48. $4 \sin^4 \theta = 1 - 2 \cos 2\theta + \cos^2 2\theta$

REVIEW PROBLEMS

The following problems review material we covered in Section 4.6. Reviewing these problems will help you with the next section.

Evaluate without using a calculator or tables.

49. $\sin \left(\arcsin \frac{3}{5}\right)$

50. $\cos \left(\arcsin \frac{3}{5}\right)$

51. $\cos (\arctan 2)$

52. $\sin (\arctan 2)$

Write an equivalent expression that involves x only.

53. $\sin (\tan^{-1} x)$

54. $\cos (\tan^{-1} x)$

55. $\tan (\sin^{-1} x)$

56. $\tan (\cos^{-1} x)$

57. Graph $y = \sin^{-1} x$

58. Graph $y = \cos^{-1} x$

SECTION 5.5 ADDITIONAL IDENTITIES

There are two main parts to this section, both of which rely on the work we have done previously with identities and formulas. In the first part of this section, we will extend our work on identities to include problems that involve inverse trigonometric functions. In the second part, we will use the formulas we obtained for the sine and cosine of a sum or difference to write some new formulas involving sums and products.

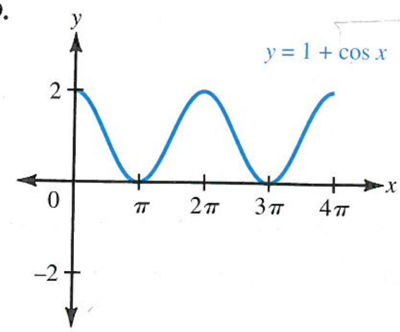
PROBLEM SET 5.4

1. $\frac{1}{2}$ **3.** 2 **5.** $-\frac{1}{\sqrt{10}}$ **7.** $-\sqrt{10}$ **9.** $\sqrt{\frac{3+2\sqrt{2}}{6}}$ **11.** $-\sqrt{\frac{3-2\sqrt{2}}{6}}$ **13.** $-3-2\sqrt{2}$

15. $\frac{2}{\sqrt{5}}$ **17.** $-\frac{7}{25}$ **19.** $-\frac{25}{7}$ **21.** $\frac{3}{\sqrt{10}}$ **23.** $\frac{7}{25}$ **25.** 0

27. See the solution to Problem 65 in Problem Set 5.3.

29.



31. $\frac{\sqrt{2+\sqrt{3}}}{2}$ **33.** $\frac{\sqrt{2+\sqrt{3}}}{2}$ **35.** $-\frac{\sqrt{2-\sqrt{3}}}{2}$

For Problems 37–47, see the Solutions Manual.

49. $\frac{3}{5}$ **51.** $\frac{1}{\sqrt{5}}$ **53.** $\frac{x}{\sqrt{x^2+1}}$ **55.** $\frac{x}{\sqrt{1-x^2}}$

57. See graph on page 222.

PROBLEM SET 5.5

1. $-\frac{1}{\sqrt{5}}$ **3.** $\frac{2\sqrt{3}-1}{2\sqrt{5}}$ **5.** $\frac{4}{5}$ **7.** $\frac{x}{\sqrt{1-x^2}}$ **9.** $2x\sqrt{1-x^2}$ **11.** $2x^2-1$

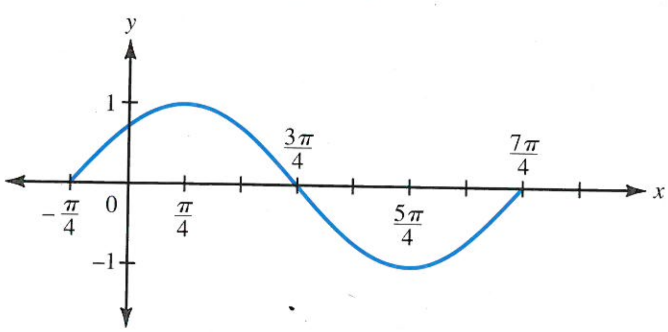
13. See the Solutions Manual. **15.** $5(\sin 8x + \sin 2x)$ **17.** $\frac{1}{2}(\cos 10x + \cos 6x)$ **19.** $\frac{1}{2}(\sin 90° + \sin 30°) = \frac{3}{4}$

21. $\frac{1}{2}(\cos 2\pi - \cos 6\pi) = 0$ **23.** See the Solutions Manual. **25.** $2 \sin 5x \cos 2x$

27. $2 \cos 30° \cos 15° = \sqrt{3} \cos 15°$ **29.** $2 \cos \frac{\pi}{3} \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$

For Problems 31–35, see the Solutions Manual.

37.



39.