SECTION 6.1 SOLVING TRIGONOMETRIC EQUATIONS

The solution set for an equation is the set of all numbers which, when used in place of the variable, make the equation a true statement. For example, the solution set for the equation $4x^2 - 9 = 0$ is $\left\{-\frac{3}{2}, \frac{3}{2}\right\}$ since these are the only two numbers that, when used in place of x, turn the equation into a true statement.

In algebra, the first kind of equations you learned to solve were linear (or first-degree) equations in one variable. Solving these equations was accomplished by applying two important properties: the *addition property of equality* and the *multiplication property of equality.* These two properties were stated as follows:

ADDITION PROPERTY OF EQUALITY

For any three algebraic expressions A, B, and C

$$\text{If } A = B$$

$$\text{then } A + C = B + C$$

In Words: Adding the same quantity to both sides of an equation will not change the solution set.

MULTIPLICATION PROPERTY OF EQUALITY

For any three algebraic expressions A, B, and C, with $C \neq 0$,

$$\text{If } A = B$$

$$\text{then } AC = BC$$

In Words: Multiplying both sides of an equation by the same nonzero quantity will not change the solution set.

Here is an example that shows how we use these two properties to solve a linear equation in one variable.

EXAMPLE 1 Solve for x: $5x + 7 = 2x - 5$.

SOLUTION

$5x + 7 = 2x - 5$	
$3x + 7 = -5$	Add $-2x$ to each side
$3x = -12$	Add -7 to each side
$x = -4$	Multiply each side by $\frac{1}{3}$

Notice in the last step we could just as easily have divided both sides by 3 instead of multiplying both sides by $\frac{1}{3}$. Division by a number and multiplication by its reciprocal are equivalent operations.

The process of solving trigonometric equations is very similar to the process of solving algebraic equations. With trigonometric equations, we look for values of an *angle* that will make the equation into a true statement. We usually begin by solving for a specific trigonometric function of that angle and then use the concepts we have developed earlier to find the angle. Here are some examples that illustrate this procedure.

EXAMPLE 2 Solve for x: $2 \sin x - 1 = 0$.

SOLUTION We can solve for $\sin x$ using our methods from algebra. We then use our knowledge of trigonometry to find x.

$$2 \sin x - 1 = 0$$
$$2 \sin x = 1$$
$$\sin x = \frac{1}{2}$$

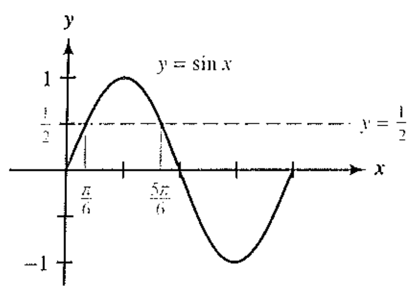


Figure 1

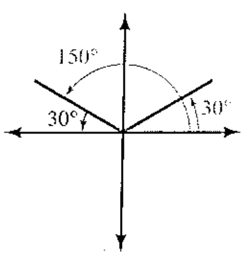


Figure 2

From Figure 1 we can see that if we are looking for radian solutions between 0 and 2π, then x is either $\pi/6$ or $5\pi/6$. On the other hand, if we want degree solutions between 0° and 360°, then our solutions will be 30° and 150°. Without the aid of Figure 1, we would reason that, since $\sin x = \frac{1}{2}$, the reference angle for x is 30°. Then, since $\frac{1}{2}$ is a positive number and the sine function is positive in quadrants I and II, x must be 30° or 150° (Figure 2).

SOLUTIONS BETWEEN 0° AND 360° OR 0 AND 2π

In Degrees			In Radians		
$x = 30°$	or	$x = 150°$	$x = \frac{\pi}{6}$	or	$x = \frac{5\pi}{6}$

Since the sine function is periodic with period 2π (or 360°), adding multiples of 2π (or 360°) will give us all solutions.

ALL SOLUTIONS (k IS AN INTEGER)

	In Degrees		In Radians
	$x = 30° + 360°k$		$x = \frac{\pi}{6} + 2k\pi$
or	$x = 150° + 360°k$	or	$x = \frac{5\pi}{6} + 2k\pi$

USING TECHNOLOGY

SOLVING EQUATIONS: FINDING ZEROS

We can solve the equation in Example 2 using a graphing calculator. If we define the left side of the equation as function Y1, then the solutions of the equation will be the x-intercepts of this function. We sometimes refer to these values as *zeros*.

Set your calculator to degree mode and define $Y_1 = 2 \sin x - 1$. Set your window variables so that

$$0 \le x \le 360, \text{ scale} = 90; \ -4 \le y \le 4, \text{ scale} = 1$$

Graph the function and use the appropriate command on your calculator to find both zeros. From Figure 3 we see that the solutions between 0° and 360° are $x = 30°$ and $x = 150°$.

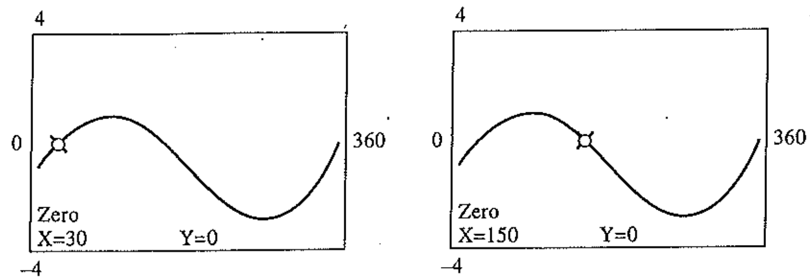


Figure 3

EXAMPLE 3 Solve $2 \sin \theta - 3 = 0$, if $0° \le \theta < 360°$.

SOLUTION We begin by solving for $\sin \theta$.

$$2 \sin \theta - 3 = 0$$

$$2 \sin \theta = 3 \quad \text{Add 3 to both sides}$$

$$\sin \theta = \frac{3}{2} \quad \text{Divide both sides by 2}$$

Since $\sin \theta$ is between -1 and 1 for all values of θ, $\sin \theta$ can never be $\frac{3}{2}$. Therefore, there is no solution to our equation.

To justify our conclusion further, we can graph $y = 2 \sin x - 3$. The graph is a sine curve with amplitude 2 that has been shifted down 3 units vertically. The graph is shown in Figure 4. Since the graph does not cross the x-axis, there is no solution to the equation $2 \sin x - 3 = 0$.

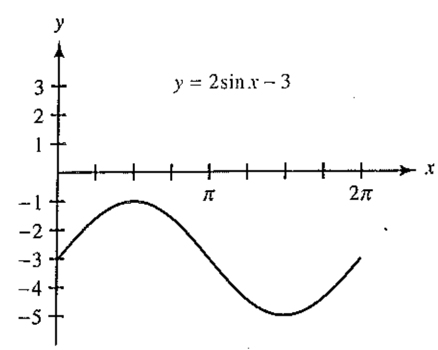


Figure 4

EXAMPLE 4 Find all degree solutions to $\sin(2A - 50°) = \dfrac{\sqrt{3}}{2}$.

SOLUTION The expression $2A - 50°$ must be coterminal with 60° or 120°, since $\sin 60° = \sqrt{3}/2$ and $\sin 120° = \sqrt{3}/2$. Therefore,

$$2A - 50° = 60° + 360°k \quad \text{or} \quad 2A - 50° = 120° + 360°k$$

where k is an integer. To solve each of these equations for A, we first add 50° to each side of the equation and then divide each side by 2. Here are the steps involved in doing so:

$$2A - 50° = 60° + 360°k \quad \text{or} \quad 2A - 50° = 120° + 360°k$$

$2A = 110° + 360°k$	$2A = 170° + 360°k$	Add 50° to each side
$A = \dfrac{110° + 360°k}{2}$	$A = \dfrac{170° + 360°k}{2}$	Divide each side by 2
$A = \dfrac{110°}{2} + \dfrac{360°k}{2}$	$A = \dfrac{170°}{2} + \dfrac{360°k}{2}$	
$A = 55° + 180°k$	$A = 85° + 180°k$	

NOTE Unless directed otherwise, let's agree to write all degree solutions to our equations in decimal degrees, to the nearest tenth of a degree.

USING TECHNOLOGY

SOLVING EQUATIONS: FINDING INTERSECTION POINTS

We can solve the equation in Example 4 with a graphing calculator by defining the expression on each side of the equation as a function. The solutions to the equation will be the x-values of the points where the two graphs intersect.

Set your calculator to degree mode and define $Y_1 = \sin(2x - 50)$ and $Y_2 = \sqrt{3}/2$. Set your window variables so that

$$0 \le x \le 360,\ \text{scale} = 90;\ -2 \le y \le 2,\ \text{scale} = 1$$

Graph both functions and use the appropriate command on your calculator to find the coordinates of the first two intersection points. From Figure 5 we see that the x-coordinates of these points are $x = 55°$ and $x = 85°$. We can also see that the next intersection points occur after 180° instead of 360°. Find the second pair of intersection points and verify that this is true.

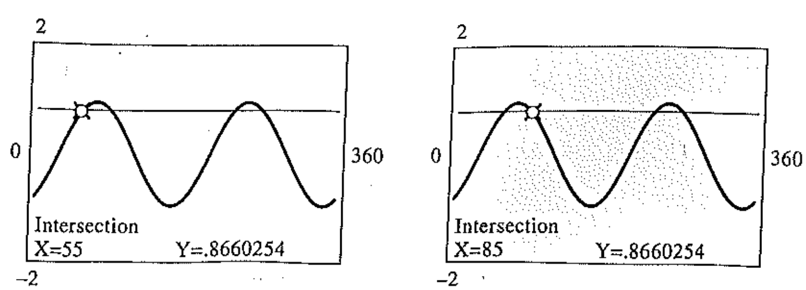


Figure 5

EXAMPLE 5 Solve $3 \sin \theta - 2 = 7 \sin \theta - 1$, if $0° \le \theta \le 360°$.

SOLUTION We can solve for $\sin \theta$ by collecting all the variable terms on the left side and all the constant terms on the right side.

$$3 \sin \theta - 2 = 7 \sin \theta - 1$$

$$-4 \sin \theta - 2 = -1 \qquad \text{Add } -7 \sin \theta \text{ to each side}$$

$$-4 \sin \theta = 1 \qquad \text{Add 2 to each side}$$

$$\sin \theta = -\frac{1}{4} \qquad \text{Divide each side by } -4$$

Since we have not memorized the angle whose sine is $-\frac{1}{4}$, we must convert $-\frac{1}{4}$ to a decimal and use a calculator to find the reference angle.

$$\hat{\theta} = \sin^{-1}(0.2500) = 14.5°$$

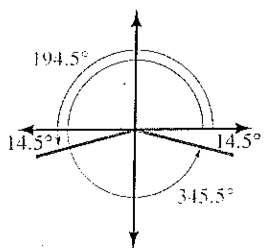


Figure 6

We find that the angle whose sine is nearest to 0.2500 is 14.5°. Therefore, the reference angle is 14.5°. Since $\sin \theta$ is negative, θ will terminate in quadrant III or IV (Figure 6).

In quadrant III we have	In quadrant IV we have
$\theta = 180° + 14.5°$	$\theta = 360° - 14.5°$
$= 194.5°$	$= 345.5°$

CALCULATOR NOTE Remember, because of the restricted values on your calculator, if you use the $\boxed{\sin^{-1}}$ key with -0.2500, your calculator will display approximately $-14.5°$, which is not within the desired interval. The best way to proceed is to find the reference angle using $\boxed{\sin^{-1}}$ with the positive value 0.2500. Then do the rest of the calculations as we have here.

The next kind of trigonometric equation we will solve is quadratic in form. In algebra, the two most common methods of solving quadratic equations are factoring and applying the quadratic formula. Here is an example that reviews the factoring method.

EXAMPLE 6 Solve $2x^2 - 9x = 5$ for x.

SOLUTION We begin by writing the equation in standard form (0 on one side—decreasing powers of the variable on the other). We then factor the left side and set each factor equal to 0.

$$2x^2 - 9x = 5$$

$$2x^2 - 9x - 5 = 0 \qquad \text{Standard form}$$

$$(2x + 1)(x - 5) = 0 \qquad \text{Factor}$$

$$2x + 1 = 0 \quad \text{or} \quad x - 5 = 0 \qquad \text{Set each factor to 0}$$

$$x = -\frac{1}{2} \quad \text{or} \quad x = 5 \qquad \text{Solving resulting equations}$$

The two solutions, $x = -\frac{1}{2}$ and $x = 5$, are the only two numbers that satisfy the original equation.

EXAMPLE 7 Solve $2\cos^2 t - 9\cos t = 5$, if $0 \le t < 2\pi$.

SOLUTION This equation is the equation from Example 6 with $\cos t$ in place of x. The fact that $0 \le t < 2\pi$ indicates we are to write our solutions in radians.

$$2\cos^2 t - 9\cos t = 5$$

$$2\cos^2 t - 9\cos t - 5 = 0 \qquad \text{Standard form}$$

$$(2\cos t + 1)(\cos t - 5) = 0 \qquad \text{Factor}$$

$$2\cos t + 1 = 0 \quad \text{or} \quad \cos t - 5 = 0 \qquad \text{Set each factor to 0}$$

$$\cos t = -\frac{1}{2} \quad \text{or} \quad \cos t = 5$$

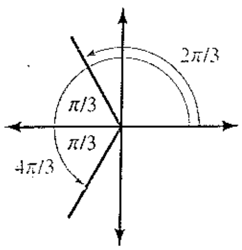


Figure 7

The first result, $\cos t = -\frac{1}{2}$, gives us a reference angle of $\hat{\theta} = \cos^{-1}(1/2) = \pi/3$. Since $\cos t$ is negative, t must terminate in quadrant II or III (Figure 7). Therefore,

$$t = \pi - \pi/3 = 2\pi/3 \quad \text{or} \quad t = \pi + \pi/3 = 4\pi/3$$

The second result, $\cos t = 5$, has no solution. For any value of t, $\cos t$ must be between -1 and 1. It can never be 5.

EXAMPLE 8 Solve $2\sin^2\theta + 2\sin\theta - 1 = 0$, if $0° \le \theta < 360°$.

SOLUTION The equation is already in standard form. If we try to factor the left side, however, we find it does not factor. We must use the quadratic formula. The quadratic formula states that the solutions to the equation

$$ax^2 + bx + c = 0$$

will be

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In our case, the coefficients a, b, and c are

$$a = 2, \quad b = 2, \quad c = -1$$

Using these numbers, we can solve for $\sin\theta$ as follows:

$$\sin\theta = \frac{-2 \pm \sqrt{4 - 4(2)(-1)}}{2(2)}$$

$$= \frac{-2 \pm \sqrt{12}}{4}$$

$$= \frac{-2 \pm 2\sqrt{3}}{4}$$

$$= \frac{-1 \pm \sqrt{3}}{2}$$

Using the approximation $\sqrt{3} = 1.7321$, we arrive at the following decimal approximations for $\sin\theta$:

$$\sin\theta = \frac{-1 + 1.7321}{2} \quad \text{or} \quad \sin\theta = \frac{-1 - 1.7321}{2}$$

$$\sin\theta = 0.3661 \quad \text{or} \quad \sin\theta = -1.3661$$

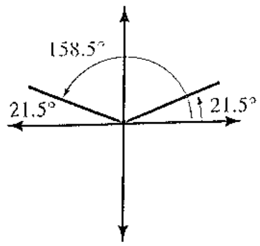


Figure 8

We will not obtain any solutions from the second expression, $\sin\theta = -1.3661$, since $\sin\theta$ must be between -1 and 1. For $\sin\theta = 0.3661$, we use a calculator to find the angle whose sine is nearest to 0.3661. That angle is 21.5°, and it is the reference angle for θ. Since $\sin\theta$ is positive, θ must terminate in quadrant I or II (Figure 8). Therefore,

$$\theta = 21.5° \quad \text{or} \quad \theta = 180° - 21.5° = 158.5°$$

GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

a. State the multiplication property of equality.

b. What is the solution set for an equation?

c. How many solutions between 0° and 360° does the equation $2\sin x - 1 = 0$ contain?

d. Under what condition is factoring part of the process of solving an equation?

PROBLEM SET 6.1

Solve each equation for θ if $0° \le \theta < 360°$. Do not use a calculator.

1. $2\sin\theta = 1$
2. $2\cos\theta = 1$
3. $2\cos\theta - \sqrt{3} = 0$
4. $2\cos\theta + \sqrt{3} = 0$
5. $2\tan\theta + 2 = 0$
6. $\sqrt{3}\cot\theta - 1 = 0$

Solve each equation for t if $0 \le t < 2\pi$. Give all answers as exact values in radians. Do not use a calculator.

7. $4\sin t - \sqrt{3} = 2\sin t$
8. $\sqrt{3} + 5\sin t = 3\sin t$
9. $2\cos t = 6\cos t - \sqrt{12}$
10. $5\cos t + \sqrt{12} = \cos t$
11. $3\sin t + 5 = -2\sin t$
12. $3\sin t + 4 = 4$

Find all solutions in the interval $0° \le \theta < 360°$. Use a calculator on the last step and write all answers to the nearest tenth of a degree.

13. $4\sin\theta - 3 = 0$
14. $4\sin\theta + 3 = 0$
15. $2\cos\theta - 5 = 3\cos\theta - 2$
16. $4\cos\theta - 1 = 3\cos\theta + 4$
17. $\sin\theta - 3 = 5\sin\theta$
18. $\sin\theta - 4 = -2\sin\theta$

Solve for x, if $0 \le x < 2\pi$. Write your answers in exact values only.

19. $(\sin x - 1)(2\sin x - 1) = 0$
20. $(\cos x - 1)(2\cos x - 1) = 0$
21. $\tan x(\tan x - 1) = 0$
22. $\tan x(\tan x + 1) = 0$
23. $\sin x + 2\sin x\cos x = 0$
24. $\cos x - 2\sin x\cos x = 0$
25. $2\sin^2 x - \sin x - 1 = 0$
26. $2\cos^2 x + \cos x - 1 = 0$

Solve for θ, if $0° \le \theta < 360°$.

27. $(2\cos\theta + \sqrt{3})(2\cos\theta + 1) = 0$
28. $(2\sin\theta - \sqrt{3})(2\sin\theta - 1) = 0$
29. $\sqrt{3}\tan\theta - 2\sin\theta\tan\theta = 0$
30. $\tan\theta - 2\cos\theta\tan\theta = 0$
31. $2\cos^2\theta + 11\cos\theta = -5$
32. $2\sin^2\theta - 7\sin\theta = -3$

Use the quadratic formula to find all solutions in the interval $0° \le \theta < 360°$ to the nearest tenth of a degree.

33. $2\sin^2\theta - 2\sin\theta - 1 = 0$
34. $2\cos^2\theta + 2\cos\theta - 1 = 0$
35. $\cos^2\theta + \cos\theta - 1 = 0$
36. $\sin^2\theta - \sin\theta - 1 = 0$
37. $2\sin^2\theta + 1 = 4\sin\theta$
38. $1 - 4\cos\theta = -2\cos^2\theta$

Write expressions representing all solutions to the equations you solved in the problems below.

39. Problem 1
40. Problem 2
41. Problem 7
42. Problem 8
43. Problem 11
44. Problem 12
45. Problem 13
46. Problem 14

Find all degree solutions to the following equations.

47. $\cos(2A - 50°) = \frac{\sqrt{3}}{2}$
48. $\sin(2A + 50°) = \frac{\sqrt{3}}{2}$
49. $\sin(3A + 30°) = \frac{1}{2}$
50. $\cos(3A + 30°) = \frac{1}{2}$
51. $\cos(4A - 20°) = -\frac{1}{2}$
52. $\sin(4A - 20°) = -\frac{1}{2}$
53. $\sin(5A + 15°) = -\frac{1}{\sqrt{2}}$
54. $\cos(5A + 15°) = -\frac{1}{\sqrt{2}}$

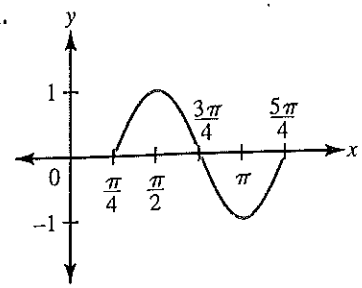
Use your graphing calculator to find the solutions to the equations you solved in the problems below by graphing the function represented by the left side of the equation and then finding its zeros. Make sure your calculator is set to degree mode.

55. Problem 3
56. Problem 4
57. Problem 13
58. Problem 14
59. Problem 29
60. Problem 30
61. Problem 33
62. Problem 34

Use your graphing calculator to find the solutions to the equations you solved in the problems below by defining the left side and right side of the equation as functions and then finding the intersection points of their graphs. Make sure your calculator is set to degree mode.

63. Problem 1
64. Problem 2
65. Problem 17
66. Problem 18
67. Problem 31
68. Problem 32
69. Problem 37
70. Problem 38
71. Problem 47
72. Problem 48
73. Problem 53
74. Problem 54

41.



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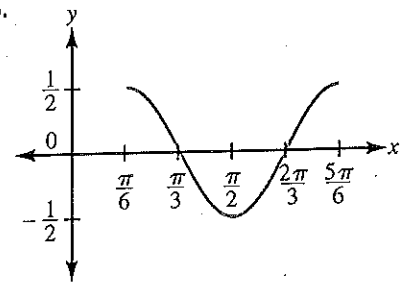
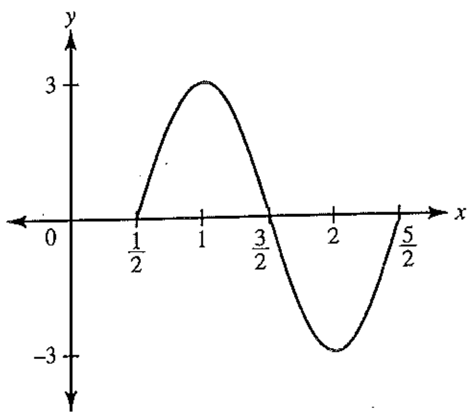


45.



CHAPTER 5 TEST

For Problems 1–12, see the Solutions Manual.

NOTE For Problems 13–18, when the equation is an identity, the proof is given in the Solutions Manual.

13. Is an identity **14.** Is an identity **15.** Not an identity **16.** Not an identity **17.** Is an identity

18. Is an identity **19.** $\frac{63}{65}$ **20.** $-\frac{56}{65}$ **21.** $-\frac{119}{169}$ **22.** $-\frac{120}{169}$ **23.** $\frac{1}{\sqrt{10}}$ **24.** $-\frac{3}{\sqrt{10}}$

NOTE For Problems 25–28, other answers are possible depending on the identity used.

25. $\frac{\sqrt{6}+\sqrt{2}}{4}$ **26.** $\frac{\sqrt{6}+\sqrt{2}}{4}$ **27.** $\frac{\sqrt{3}-1}{\sqrt{3}+1}$ **28.** $\frac{\sqrt{3}+1}{\sqrt{3}-1}$ **29.** $\cos 9x$ **30.** $\sin 90° = 1$

31. $\frac{3}{5}, -\sqrt{\frac{5-2\sqrt{5}}{10}}$ **32.** $\frac{3}{5}, \sqrt{\frac{10-\sqrt{10}}{20}}$ **33.** 1 **34.** $\pm\frac{\sqrt{3}}{2}$ **35.** $\frac{11}{5\sqrt{5}}$ **36.** $\frac{11}{5\sqrt{5}}$ **37.** $1 - 2x^2$

38. $2x\sqrt{1-x^2}$ **39.** $\frac{1}{2}(\cos 2x - \cos 10x)$ **40.** $2\cos 45° \cos(-30°) = \frac{\sqrt{6}}{2}$

CHAPTER 6

PROBLEM SET 6.1

1. 30°, 150° **3.** 30°, 330° **5.** 135°, 315° **7.** $\pi/3, 2\pi/3$ **9.** $\frac{\pi}{6}, \frac{11\pi}{6}$ **11.** $\frac{3\pi}{2}$ **13.** 48.6°, 131.4°

15. $\varnothing$ **17.** 228.6°, 311.4° **19.** $\frac{\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$ **21.** $0, \frac{\pi}{4}, \pi, \frac{5\pi}{4}$ **23.** $0, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}$ **25.** $\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$

27. 120°, 150°, 210°, 240° **29.** 0°, 60°, 180°, 120° **31.** 120°, 240° **33.** 201.5°, 338.5° **35.** 51.8°, 308.2°

37. 17.0°, 163.0° **39.** 30° + 360°k, 150° + 360°k **41.** $\frac{\pi}{3} + 2k\pi, \frac{2\pi}{3} + 2k\pi$ **43.** $\frac{3\pi}{2} + 2k\pi$
45. 48.6° + 360°k, 131.4° + 360°k **47.** 40° + 180°k, 190° + 180°k **49.** 120°k, 40° + 120°k
51. 35° + 90°k, 65° + 90°k **53.** 42° + 72°k, 60° + 72°k

For Problems 55–73, see the answer for the corresponding problem.
75. $h = -16t^2 + 750t$ **77.** 1,436 ft **79.** 15.7° **81.** $\sin 2A = 2 \sin A \cos A$ **83.** $\cos 2A = 2\cos^2 A - 1$
85. $\frac{1}{\sqrt{2}} \sin\theta + \frac{1}{\sqrt{2}} \cos\theta$ **87.** $\frac{\sqrt{6} + \sqrt{2}}{4}$ **89.** See the Solutions Manual.

PROBLEM SET 6.2

1. 30°, 330° **3.** 225°, 315° **5.** 45°, 135°, 225°, 315° **7.** 30°, 150° **9.** 30°, 90°, 150°, 270°
11. 60°, 180°, 300° **13.** $\frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$ **15.** $0, \frac{2\pi}{3}, \frac{4\pi}{3}$ **17.** $\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$ **19.** $\frac{\pi}{3}, \frac{5\pi}{3}$
21. $\frac{2\pi}{3}, \frac{4\pi}{3}$ **23.** $\frac{\pi}{4}$ **25.** 30°, 90° **27.** 60°, 180° **29.** 60°, 300° **31.** 120°, 180° **33.** 210°, 330°
35. 36.9°, 48.2°, 311.8°, 323.1° **37.** 36.9°, 143.1°, 216.9°, 323.1° **39.** 225° + 360°k, 315° + 360°k
41. $\frac{\pi}{4} + 2k\pi$ **43.** 120° + 360°k, 180° + 360°k **45.** See the Solutions Manual. **47.** 68.5°, 291.5°
49. 218.2°, 321.8° **51.** 73.0°, 287.0° **53.** 0.3630, 2.1351 **55.** 3.4492, 5.9756 **57.** 0.3166, 1.9917
59. $\sqrt{\frac{3 - \sqrt{5}}{6}}$ **61.** $\sqrt{\frac{6}{3 - \sqrt{5}}}$ **63.** $\sqrt{\frac{3 - \sqrt{5}}{3 + \sqrt{5}}}$ or $\frac{3 - \sqrt{5}}{2}$
65. See the solution to Problem 65 in Problem Set 5.3. **67.** $\frac{\sqrt{2 - \sqrt{2}}}{2}$

PROBLEM SET 6.3

1. 30°, 60°, 210°, 240° **3.** 67.5°, 157.5°, 247.5°, 337.5° **5.** 60°, 180°, 300° **7.** $\frac{\pi}{8}, \frac{3\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}$
9. $\frac{\pi}{3}, \pi, \frac{5\pi}{3}$ **11.** $\frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$ **13.** 15° + 180°k, 75° + 180°k **15.** 30° + 120°k, 90° + 120°k
17. 6° + 36°k, 12° + 36°k **19.** 112.5°, 157.5°, 292.5°, 337.5° **21.** 20°, 100°, 140°, 220°, 260°, 340°
23. 15°, 105°, 195°, 285° **25.** $\frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}, \frac{25\pi}{18}, \frac{29\pi}{18}$ **27.** $\frac{5\pi}{18}, \frac{7\pi}{18}, \frac{17\pi}{18}, \frac{19\pi}{18}, \frac{29\pi}{18}, \frac{31\pi}{18}$
29. $\frac{\pi}{10} + \frac{2k\pi}{5}$ **31.** $\frac{\pi}{8} + \frac{k\pi}{2}, \frac{3\pi}{8} + \frac{k\pi}{2}$ **33.** $\frac{\pi}{5} + \frac{2k\pi}{5}$ **35.** 10° + 120°k, 50° + 120°k, 90° + 120°k
37. 60° + 180°k, 90° + 180°k, 120° + 180°k **39.** 20° + 60°k, 40° + 60°k **41.** 0°, 270° **43.** 180°, 270°
45. 96.8°, 173.2°, 276.8°, 353.2° **47.** 27.4°, 92.6°, 147.4°, 212.6°, 267.4°, 332.6°
49. 50.4°, 84.6°, 140.4°, 174.6°, 230.4°, 264.6°, 320.4°, 354.6°
51. 4.0 min and 16.0 min **53.** 6 **55.** $\frac{1}{4}$ second (and every second after that) **57.** $\frac{1}{12}$

For Problems 59–63, see the Solutions Manual.
65. $-\frac{4\sqrt{2}}{9}$ **67.** $\sqrt{\frac{3 - 2\sqrt{2}}{6}}$ **69.** $\frac{4 - 6\sqrt{2}}{15}$ **71.** $\frac{15}{4 - 6\sqrt{2}}$