89. $\dfrac{1-\tan^2 x}{1+\tan^2 x}=\dfrac{1-\dfrac{\sin^2 x}{\cos^2 x}}{1+\dfrac{\sin^2 x}{\cos^2 x}}$ Ratio identity

$=\dfrac{\cos^2 x-\sin^2 x}{\cos^2 x+\sin^2 x}$ Multiply numerator and denominator by $\cos^2 x$

$=\dfrac{\cos 2x}{1}$ Double-angle identity and Pythagorean identity

$=\cos 2x$ Simplify

Problem Set 6.2

1.

$$\sqrt{3}\sec\theta=2$$
$$\sec\theta=\frac{2}{\sqrt{3}}$$
$$\cos\theta=\frac{\sqrt{3}}{2}$$

$\hat{\theta}=30^\circ$ and θ is in QI or QIV

$\theta=30^\circ$ or 330°

3.

$$\sqrt{2}\csc\theta+5=3$$
$$\sqrt{2}\csc\theta=-2$$
$$\csc\theta=-\frac{2}{\sqrt{2}}$$
$$\sin\theta=-\frac{\sqrt{2}}{2}$$

$\hat{\theta}=45^\circ$ and θ is in QIII or QIV

$\theta=225^\circ$ or 315°

5.

$4\sin\theta-2\csc\theta=0$

$4\sin\theta-\dfrac{2}{\sin\theta}=0$ $\csc\theta=\dfrac{1}{\sin\theta}$

$4\sin^2\theta-2=0$ Multiply both sides by $\sin\theta$

$4\sin^2\theta=2$ Add 2 to both sides

$\sin^2\theta=\dfrac{1}{2}$ Divide both sides by 4

$\sin\theta=\pm\dfrac{1}{\sqrt{2}}$ Take square root of both sides

$\hat{\theta}=45^\circ$ and θ is in QI, QII, QIII, or QIV

$\theta=45^\circ, 135^\circ, 225^\circ, 315^\circ$

7.
$$\sec\theta - 2\tan\theta = 0$$

$$\frac{1}{\cos\theta} - 2\left(\frac{\sin\theta}{\cos\theta}\right) = 0$$ Reciprocal and ratio identities

$$1 - 2\sin\theta = 0$$ Multiply both sides by $\cos\theta$ $(\cos\theta \neq 0)$

$$-2\sin\theta = -1$$ Solve equation and check

$$\sin\theta = \frac{1}{2}$$

$$\theta = 30^\circ \text{ or } 150^\circ$$

9.
$$\sin 2\theta - \cos\theta = 0$$

$$2\sin\theta\cos\theta - \cos\theta = 0$$ Double-angle identity

$$\cos\theta(2\sin\theta - 1) = 0$$ Factor out $\cos\theta$

$$\cos\theta = 0 \quad \text{or} \quad 2\sin\theta - 1 = 0$$ Set each factor = 0

$$\theta = 90^\circ, 270^\circ \quad 2\sin\theta = 1$$ Solve each equation

$$\sin\theta = \frac{1}{2}$$

$$\theta = 30^\circ \text{ or } 150^\circ$$

11.
$$2\cos\theta + 1 = \sec\theta$$

$$2\cos\theta + 1 = \frac{1}{\cos\theta}$$ Reciprocal identity

$$2\cos^2\theta + \cos\theta = 1$$ Multiply both sides by $\cos\theta$ $(\cos\theta \neq 0)$

$$2\cos^2\theta + \cos\theta - 1 = 0$$

$$(2\cos\theta - 1)(\cos + 1) = 0$$ Factor

$$2\cos\theta - 1 = 0 \quad \text{or} \quad \cos\theta + 1 = 0$$ Set each factor = 0

$$2\cos\theta = 1 \qquad \cos\theta = -1$$ Solve each equation and check

$$\cos\theta = \frac{1}{2} \qquad \theta = 180^\circ$$

$$\theta = 60^\circ \text{ or } 300^\circ$$

13.
$$\cos 2x - 3\sin x - 2 = 0$$

$$1 - 2\sin^2 x - 3\sin x - 2 = 0$$ Double-angle identity

$$2\sin^2 x + 3\sin x + 1 = 0$$ Multiply both sides by -1 and simplify

$$(2\sin x + 1)(\sin x + 1) = 0$$ Factor

$$2\sin x + 1 = 0 \quad \text{or} \quad \sin x + 1 = 0$$ Set each factor = 0

$$2\sin x = -1 \qquad \sin x = -1$$

$$\sin x = -\frac{1}{2} \qquad x = \frac{3\pi}{2}$$ Solve each equation

$$x = \frac{7\pi}{6} \text{ or } \frac{11\pi}{6}$$

15.

$$\cos x - \cos 2x = 0$$

$$\cos x - \left(2\cos^2 x - 1\right) = 0 \quad \text{Double-angle identity}$$

$$\cos x - 2\cos^2 x + 1 = 0 \quad \text{Simplify}$$

$$2\cos^2 x - \cos x - 1 = 0 \quad \text{Multiply both sides by } -1$$

$$\left(2\cos x + 1\right)\left(\cos x - 1\right) = 0 \quad \text{Factor}$$

$$2\cos x + 1 = 0 \quad \text{or} \quad \cos x - 1 = 0 \quad \text{Set each factor} = 0$$

$$2\cos x = -1 \qquad \cos x = 1 \quad \text{Solve each equation}$$

$$\cos x = -\frac{1}{2} \qquad x = 0$$

$$\hat{x} = \frac{\pi}{3}$$

$$x = \frac{2\pi}{3} \quad \text{or} \quad \frac{4\pi}{3}$$

17.

$$2\cos^2 x + \sin x - 1 = 0$$

$$2\left(1 - \sin^2 x\right) + \sin x - 1 = 0 \quad \text{Pythagorean identity}$$

$$2 - 2\sin^2 x + \sin x - 1 = 0 \quad \text{Simplify}$$

$$2\sin^2 x - \sin x - 1 = 0 \quad \text{Multiply both sides by } -1 \text{ and simplify}$$

$$\left(2\sin x + 1\right)\left(\sin x - 1\right) = 0 \quad \text{Factor}$$

$$2\sin x + 1 = 0 \quad \text{or} \quad \sin x - 1 = 0 \quad \text{Set each factor} = 0$$

$$2\sin x = -1 \qquad \sin x = 1 \quad \text{Solve each equation}$$

$$\sin x = -\frac{1}{2} \qquad x = \frac{\pi}{2}$$

$$x = \frac{7\pi}{6} \quad \text{or} \quad \frac{11\pi}{6}$$

19.

$$4\sin^2 x + 4\cos x - 5 = 0$$

$$4\left(1 - \cos^2 x\right) + 4\cos x - 5 = 0 \quad \text{Pythagorean identity}$$

$$4 - 4\cos^2 x + 4\cos x - 5 = 0 \quad \text{Simplify}$$

$$-4\cos^2 x + 4\cos x - 1 = 0 \quad \text{Simplify}$$

$$4\cos^2 x - 4\cos\theta + 1 = 0 \quad \text{Multiply both sides by } -1$$

$$\left(2\cos x - 1\right)\left(2\cos x - 1\right) = 0 \quad \text{Factor}$$

$$2\cos x - 1 = 0 \quad \text{or} \quad 2\cos x - 1 = 0 \quad \text{Set each factor} = 0$$

$$2\cos x = 1 \quad \text{Solve the equation}$$

$$\cos x = \frac{1}{2}$$

$$x = \frac{\pi}{3} \quad \text{or} \quad \frac{5\pi}{3}$$

21.

$$2\sin x + \cot x - \csc x = 0$$

$$2\sin x + \frac{\cos x}{\sin x} - \frac{1}{\sin x} = 0 \qquad \text{Ratio and reciprocal identities}$$

$$2\sin^2 x + \cos x - 1 = 0 \qquad \text{Multiply both sides by } \sin x \left(\sin x \neq 0\right)$$

$$2\left(1 - \cos^2 x\right) + \cos x - 1 = 0 \qquad \text{Pythagorean identity}$$

$$2 - 2\cos^2 x + \cos x - 1 = 0 \qquad \text{Simplify}$$

$$2\cos^2 x - \cos x - 1 = 0 \qquad \text{Multiply both sides by } -1 \text{ and simplify}$$

$$\left(2\cos x + 1\right)\left(\cos x - 1\right) = 0 \qquad \text{Factor}$$

$$2\cos x + 1 = 0 \quad \text{or} \quad \cos x - 1 = 0 \qquad \text{Set each factor} = 0$$

$$2\cos x = -1 \qquad \cos x = 1 \qquad \text{Solve each equation and check}$$

$$\cos x = -\frac{1}{2} \qquad x = 0 \text{ which is not possible because } \sin x \neq 0$$

$$x = \frac{2\pi}{3} \text{ or } \frac{4\pi}{3}$$

23.

$$\sin x + \cos x = \sqrt{2}$$

$$\sin^2 x + 2\sin x\cos x + \cos^2 x = 2 \qquad \text{Square both sides}$$

$$2\sin x\cos x + 1 = 2 \qquad \text{Pythagorean identity}$$

$$\sin 2x = 1 \qquad \text{Double-angle identity}$$

$$2x = \frac{\pi}{2} \text{ or } 2x = \frac{5\pi}{2} \qquad \text{Solve equation and check}$$

$$x = \frac{\pi}{4} \qquad x = \frac{5\pi}{4} \qquad \text{Possible solutions}$$

Check each possible solution:

$$\sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}$$

$$= \frac{2\sqrt{2}}{2}$$

$$= \sqrt{2} \qquad \text{It checks}$$

$$\sin\frac{5\pi}{4} + \cos\frac{5\pi}{4} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}$$

$$= 0 \qquad \text{Does not check}$$

The answer is $\frac{\pi}{4}$ only.

25. $\sqrt{3}\sin\theta+\cos\theta=\sqrt{3}$

$\cos\theta=\sqrt{3}-\sqrt{3}\sin\theta$	Subtract $\sqrt{3}\sin\theta$ from both sides
$\cos\theta=\sqrt{3}(1-\sin\theta)$	Factor out $\sqrt{3}$
$\cos^2\theta=3(1-2\sin\theta+\sin^2\theta)$	Square both sides
$1-\sin^2\theta=3-6\sin\theta+3\sin^2\theta$	Pythagorean identity
$4\sin^2\theta-6\sin\theta+2=0$	Simplify
$2\sin^2\theta-3\sin\theta+1=0$	Divide both sides by 2
$(2\sin\theta-1)(\sin\theta-1)=0$	Factor
$2\sin\theta-1=0$ or $\sin\theta-1=0$	Set each factor = 0
$2\sin\theta=1$ $\quad\sin\theta=1$	Solve each equation and check
$\sin\theta=\frac{1}{2}$ $\quad\theta=90^\circ$	Possible solutions
$\theta=30^\circ$ or 150°	

Check each possible solution:

$$\sqrt{3}\sin 30^\circ+\cos 30^\circ=\sqrt{3}\left(\frac{1}{2}\right)+\frac{\sqrt{3}}{2}$$

$$=\sqrt{3}\quad\text{It checks}$$

$$\sqrt{3}\sin 150^\circ=\cos 150^\circ=\sqrt{3}\left(\frac{1}{2}\right)+\left(-\frac{\sqrt{3}}{2}\right)$$

$$=0\quad\text{Does not check}$$

$$\sqrt{3}\sin 90^\circ+\cos 90^\circ=\sqrt{3}(1)+0$$

$$=\sqrt{3}\quad\text{It checks}$$

Answers: 30° or 90°

27. $\sqrt{3}\sin\theta-\cos\theta=1$

$\sqrt{3}\sin\theta=\cos\theta+1$	Isolate $\sqrt{3}\sin\theta$
$3\sin^2\theta=\cos^2\theta+2\cos\theta+1$	Square both sides
$3(1-\cos^2\theta)=\cos^2\theta+2\cos\theta+1$	Pythagorean identity
$3-3\cos^2\theta=\cos^2\theta+2\cos\theta+1$	Simplify
$4\cos^2\theta+2\cos\theta-2=0$	Put in standard form
$2(2\cos\theta-1)(\cos\theta+1)=0$	Factor
$2\cos\theta-1=0$ or $\cos\theta+1=0$	Set each factor = 0
$2\cos\theta=1$ $\quad\cos\theta=-1$	Solve each equation and check
$\cos\theta=\frac{1}{2}$ $\quad\theta=180^\circ$	
$\theta=60^\circ$ or 300°	Possible solutions

The solution is 60° or 180°. (300° does not check.)

29.

$$\sin\frac{\theta}{2}-\cos\theta=0$$

$\sin\frac{\theta}{2}=\cos\theta$	Add $\cos\theta$ to both sides
$\sin^2\frac{\theta}{2}=\cos^2\theta$	Square both sides
$\frac{1-\cos\theta}{2}=\cos^2\theta$	$\sin\frac{\theta}{2}=\pm\sqrt{\frac{1-\cos\theta}{2}}$
$1-\cos\theta=2\cos^2\theta$	Multiply both sides by 2
$2\cos^2\theta+\cos\theta-1=0$	Rewrite in standard form
$(2\cos\theta-1)(\cos\theta+1)=0$	Factor
$2\cos\theta-1=0$ or $\cos\theta+1=0$	Set each factor = 0
$2\cos\theta=1$ $\quad$ $\cos\theta=-1$	Solve each equation and check
$\cos\theta=\frac{1}{2}$ $\quad$ $\theta=180^\circ$	Possible solutions
$\theta=60^\circ$ or 300°	

The solution is 60° or 300°. (180° does not check.)

31.

$$\cos\frac{\theta}{2}-\cos\theta=1$$

$\pm\sqrt{\frac{1+\cos\theta}{2}}-\cos\theta=1$	Half-angle identity
$\pm\sqrt{\frac{1+\cos\theta}{2}}=\cos\theta+1$	Add $\cos\theta$ to both sides
$\frac{1+\cos\theta}{2}=\cos^2\theta+2\cos\theta+1$	Square both sides
$1+\cos\theta=2\cos^2\theta+4\cos\theta+2$	Multiply both sides by 2
$0=2\cos^2\theta+3\cos\theta+1$	Put in standard form
$0=(2\cos\theta+1)(\cos\theta+1)$	Factor
$2\cos\theta+1=0$ or $\cos\theta+1=0$	Set each factor = 0
$2\cos\theta=-1$ $\quad$ $\cos\theta=-1$	Solve each equation and check
$\cos\theta=-\frac{1}{2}$ $\quad$ $\theta=180^\circ$	Possible solutions
$\theta=120^\circ$ or 240°	

The solution is 120° or 180°. (240° does not check.)

33.

$6\cos\theta + 7\tan\theta = \sec\theta$	
$6\cos\theta + \dfrac{7\sin\theta}{\cos\theta} = \dfrac{1}{\cos\theta}$	Ratio and reciprocal identities
$6\cos^2\theta + 7\sin\theta = 1$	Multiply both sides by $\cos\theta\ (\cos\theta \neq 0)$
$6(1-\sin^2\theta) + 7\sin\theta = 1$	Pythagorean identity
$6 - 6\sin^2\theta + 7\sin^2\theta = 1$	Simplify
$-6\sin^2\theta + 7\sin\theta + 5 = 0$	Subtract 1 from both sides
$6\sin^2\theta - 7\sin\theta - 5 = 0$	Multiply both sides by -1
$(3\sin\theta - 5)(2\sin\theta + 1) = 0$	Factor
$3\sin\theta - 5 = 0$ or $2\sin\theta + 1 = 0$	Set each factor = 0
$3\sin\theta = 5$ $2\sin\theta = -1$	Solve each equation and check
$\sin\theta = \dfrac{5}{3}$ $\sin\theta = -\dfrac{1}{2}$	
No solution $\theta = 210^\circ$ or 330°	Both check

35.

$23\csc^2\theta - 22\cot\theta\csc\theta - 15 = 0$	
$23\left(\dfrac{1}{\sin^2\theta}\right) - 22\left(\dfrac{\cos\theta}{\sin\theta}\right)\left(\dfrac{1}{\sin\theta}\right) - 15 = 0$	Reciprocal and ratio identities
$\dfrac{23}{\sin^2\theta} - \dfrac{22\cos\theta}{\sin^2\theta} - 15 = 0$	Simplify
$23 - 22\cos\theta - 15\sin^2\theta = 0$	Multiply both sides by $\sin^2\theta$
$23 - 22\cos\theta - 15(1-\cos^2\theta) = 0$	Pythagorean identity
$23 - 22\cos\theta - 15 + 15\cos^2\theta = 0$	Simplify
$15\cos^2\theta - 22\cos^2\theta + 8 = 0$	Put in standard form
$(5\cos\theta - 4)(3\cos\theta - 2) = 0$	Factor
$5\cos\theta - 4 = 0$ or $3\cos\theta - 2 = 0$	Set each factor = 0
$5\cos\theta = 4$ $3\cos\theta = 2$	Solve each equations and check
$\cos\theta = \dfrac{4}{5}$ $\cos\theta = \dfrac{2}{3}$	
$\theta = 36.9^\circ$ or 323.1° $\theta = 48.2^\circ$ or 311.8°	All check

37.

$7\sin^2\theta - 9\cos 2\theta = 0$	
$7\sin^2\theta - 9(1 - 2\sin^2\theta) = 0$	Double-angle identity
$7\sin^2\theta - 9 + 18\sin^2\theta = 0$	Simplify left side
$25\sin^2\theta = 9$	Add 9 to both sides
$\sin^2\theta = \dfrac{9}{25}$	Divide both sides by 25

This problem is continued o the next page

$$\sin\theta = \pm\frac{3}{5} \qquad \text{Take square root of both sides}$$

$$\hat{\theta} = 36.9^\circ \text{ and } \theta = \text{QI, QII, QIII, or QIV}$$

$$\theta = 36.9^\circ, 143.1^\circ, 216.9^\circ \text{ or } 323.1^\circ$$

39. In problem 3, we get 225° or 315°. All solutions would be $225^\circ + 360^\circ k$ or $315^\circ + 360^\circ k$.

41. In problem 23 we get $x = \frac{\pi}{4}$. All solutions would be $\frac{\pi}{4} + 2k\pi$.

43. In problem 31, we get 120° or 180°. All solutions would be $120^\circ + 360^\circ k$ or $180^\circ + 360^\circ k$.

45.
$$r^4\csc^2\theta - R^4\csc\theta\cot\theta = r^4\cdot\frac{1}{\sin^2\theta} - R^4\cdot\frac{1}{\sin\theta}\cdot\frac{\cos\theta}{\sin\theta}$$
$$= \frac{r^4}{\sin^2\theta} - \frac{R^4\cos\theta}{\sin^2\theta}$$
$$= \frac{r^4 - R^4\cos\theta}{\sin^2\theta}$$

This expression is zero only when the numerator is zero. Therefore,

$$r^4 - R^4\cos\theta = 0$$
$$-R^4\cos\theta = -r^4$$
$$\cos\theta = \frac{r^4}{R^4}$$

Therefore, when $\cos\theta = \frac{r^4}{R^4}$, then $r^4\csc^2\theta - R^4\csc\theta\cot\theta = 0$

47.

$2\sin^2\theta - 2\cos\theta - 1 = 0$	
$2(1-\cos^2\theta) - 2\cos\theta - 1 = 0$	Pythagorean identity
$2 - 2\cos^2\theta - 2\cos\theta - 1 = 0$	Simplify
$-2\cos^2\theta - 2\cos\theta + 1 = 0$	Put in standard form
$2\cos^2\theta + 2\cos\theta - 1 = 0$	Multiply both sides by -1

This problem is continued on the next page

We apply the quadratic formula with $a = 2$, $b = 2$, and $c = -1$:

$$\cos\theta = \frac{-2 \pm \sqrt{2^2 - 4(2)(-1)}}{2(2)}$$

$$= \frac{-2 \pm \sqrt{12}}{4}$$

$$\cos\theta = \frac{-2 + 3.464}{4} \quad \text{or} \quad \cos\theta = \frac{-2 - 3.464}{4}$$

$$= 0.366 \qquad\qquad = -1.366$$

$\hat{\theta} = 68.5^\circ$ and θ in in QI, QIV — No solution

$\theta = 68.5^\circ$ or 291.5°

49.

$$\cos^2\theta + \sin\theta = 0$$

$1 - \sin^2\theta + \sin\theta = 0$ — Pythagorean identity

$-\sin^2\theta + \sin\theta + 1 = 0$ — Put in standard form

$\sin^2\theta - \sin\theta - 1 = 0$ — Multiply both sides by -1

We apply the quadratic formula with $a = 1$, $b = -1$, and $c = -1$:

$$\sin\theta = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{1 \pm \sqrt{5}}{2}$$

$$\sin\theta = \frac{1 + 2.236}{2} \quad \text{or} \quad \sin\theta = \frac{1 - 2.236}{2}$$

$$= 1.618 \qquad\qquad = -0.618$$

No solution — $\hat{\theta} = 38.2^\circ$ and θ in in QIII, QIV

$\theta = 218.2^\circ$ or 321.8°

51.

$$2\sin^2\theta = 3 - 4\cos\theta$$

$2(1 - \cos^2\theta) = 3 - 4\cos\theta$ — Pythagorean identity

$2 - 2\cos^2\theta = 3 - 4\cos\theta$ — Simplify

$-2\cos^2\theta + 4\cos\theta - 1 = 0$ — Put in standard form

$2\cos^2\theta - 4\cos\theta + 1 = 0$ — Multiply both sides by -1

We apply the quadratic formula with $a = 2$, $b = -4$, and $c = 1$:

$$\cos\theta = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(1)}}{2(2)}$$

$$= \frac{4 \pm \sqrt{8}}{4}$$

This problem is continued on the next page

$$\cos\theta = \frac{4+2.828}{4} = 1.707 \quad \text{or} \quad \cos\theta = \frac{4-2.828}{4} = 0.293$$

No solution

$\hat{\theta} = 73.0^\circ$ and θ is in QI and QIV

$\theta = 73.0^\circ$ or 287.0°

53. Graph $y_1 = \cos(x) + 3\sin(x) - 2$ on your graphing calculator. Set the window for x between 0 and 6.28 and y between -1 and 1. Use the zero or root finder to locate the zeros. The solution is 0.3630 or 2.1351.

55. Graph $y_1 = (\sin(x))^2 - 3\sin(x) - 1$ on your graphing calculator. The window settings are the same as problem 53. Use the zero or root finder to locate the zeros. The solution is 3.4492 or 5.9756.

57. Graph $y_1 = \frac{1}{\cos(x)} + 2 - \frac{1}{\tan(x)}$ on your graphing calculator. The window settings are the same as problem 53. Use the zero or root finder to locate the zeros. The solution is 0.3166 or 1.9917.

59. $\cos A = \sqrt{1-\sin^2 A}$ with A in QI

$$= \sqrt{1-\left(\frac{2}{3}\right)^2} = \sqrt{1-\frac{4}{9}} = \sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{3}$$

$$\sin\frac{A}{2} = \sqrt{\frac{1-\cos A}{2}} = \sqrt{\frac{1-\sqrt{5}/3}{2}} = \sqrt{\frac{3-\sqrt{5}}{6}}$$

61.

$$\csc\frac{A}{2} = \frac{1}{\sin\frac{A}{2}}$$

$$= \frac{1}{\sqrt{\frac{3-\sqrt{5}}{6}}} \quad \text{(from problem 59)}$$

$$= \sqrt{\frac{6}{3-\sqrt{5}}}$$

63. $\tan\frac{A}{2}=\dfrac{\sin\frac{A}{2}}{\cos\frac{A}{2}}$

We know that $\cos A=\frac{\sqrt{5}}{3}$ and $\sin\frac{A}{2}=\sqrt{\frac{3-\sqrt{5}}{6}}$ from problem 59.

$$\cos\frac{A}{2}=\sqrt{\frac{1+\cos A}{2}}$$

$$=\sqrt{\frac{1+\sqrt{5}/3}{2}}$$

$$=\sqrt{\frac{3+\sqrt{5}}{6}}$$

Therefore, $\tan\frac{A}{2}=\dfrac{\sqrt{\frac{3-\sqrt{5}}{6}}}{\sqrt{\frac{3+\sqrt{5}}{6}}}$

$$=\sqrt{\frac{3-\sqrt{5}}{3+\sqrt{5}}}$$

$=\frac{3-\sqrt{5}}{2}$ Rationalize the denominator

67. $\sin 22.5^\circ=\sin\frac{1}{2}\left(45^\circ\right)$

$$=\sqrt{\frac{1-\cos 45^\circ}{2}}$$

$$=\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}}$$

$$=\sqrt{\frac{2-\sqrt{2}}{4}}$$

$$=\frac{\sqrt{2-\sqrt{2}}}{2}$$