

## Worksheet – Binomial expansion

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(from PurpleMath.com)

The Binomial Theorem is a quick way (okay, it's a less slow way) of expanding (or multiplying out) a binomial expression that has been raised to some (generally inconveniently large) power. For instance, the expression  $(3x - 2)^{10}$  would be very painful to multiply out by hand. Thankfully, somebody figured out a formula for this expansion, and we can plug the binomial  $3x - 2$  and the power 10 into that formula to get that expanded (multiplied-out) form.

The formal expression of the Binomial Theorem is as follows:

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$$

The parenthetical bit above has these equivalents:

$$\binom{n}{k} = {}_n C_k = \frac{n!}{(n-k)!k!}$$

Recall that the [factorial notation](#) "n!" means "the product of all the whole numbers between 1 and n", so, for instance,  $6! = 1 \times 2 \times 3 \times 4 \times 5 \times 6$ . Then the notation " ${}_{10}C_7$ " (often pronounced as "ten, choose seven") means:

$$\begin{aligned} {}_{10}C_7 &= \frac{10!}{(10-7)!7!} = \frac{10!}{3!7!} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10}{1 \cdot 2 \cdot 3 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} \\ &= \frac{\cancel{1} \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{4} \cdot \cancel{5} \cdot \cancel{6} \cdot \cancel{7} \cdot 8 \cdot 9 \cdot 10}{\cancel{1} \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{1} \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{4} \cdot \cancel{5} \cdot \cancel{6} \cdot \cancel{7}} = 4 \cdot 3 \cdot 10 = 120 \end{aligned}$$

Many calculators can evaluate this "*n* choose *m*" notation for you. Just look for a key that looks like "*n**C**m*" or "*n**C**r*", or for a similar item on the "Prob" or "Math" menu, or check your owner's manual under "probability" or "combinations".

```
MATH NUM CPX MODE
1:rand
2:nPr
3:nCr
4:|
5:randInt(
6:randNorm(
7:randBin(
```

The evaluation will probably look something like this:



I could never remember the formula for the Binomial Theorem, so instead, I just learned how it worked. I noticed that the powers on each term in the expansion always added up to whatever  $n$  was, and that the terms counted up from zero to  $n$ . Returning to our initial example of  $(3x - 2)^{10}$ , the powers on every term of the expansion will add up to 10, and the powers on the terms will increment by counting up from zero to 10:

$$\begin{aligned}
 (3x - 2)^{10} &= {}_{10}C_0 (3x)^{10-0}(-2)^0 + {}_{10}C_1 (3x)^{10-1}(-2)^1 + {}_{10}C_2 (3x)^{10-2}(-2)^2 \\
 &+ {}_{10}C_3 (3x)^{10-3}(-2)^3 + {}_{10}C_4 (3x)^{10-4}(-2)^4 + {}_{10}C_5 (3x)^{10-5}(-2)^5 \\
 &+ {}_{10}C_6 (3x)^{10-6}(-2)^6 + {}_{10}C_7 (3x)^{10-7}(-2)^7 + {}_{10}C_8 (3x)^{10-8}(-2)^8 \\
 &+ {}_{10}C_9 (3x)^{10-9}(-2)^9 + {}_{10}C_{10} (3x)^{10-10}(-2)^{10}
 \end{aligned}$$

Note how the highlighted counter number counts up from zero to 10, with the factors on the ends of each term having the counter number, and the factor in the middle having the counter number subtracted from 10. This pattern is all you really need to know about the Binomial Theorem; this pattern is how it works.

Your first step, given a binomial to expand, should be to plug it into the Theorem, just like I did above. Don't try to do too many steps at once. Only after you've set up your binomial in the Theorem's pattern should you start to simplify the terms. The Binomial Theorem works best as a "plug-n-chug" process, but you should plug in first; chug later. I've done my "plugging" above; now "chugging" gives me:

$$\begin{aligned}
 &(1)(59049)x^{10}(1) + (10)(19683)x^9(-2) + (45)(6561)x^8(4) + (120)(2187)x^7(-8) \\
 &+ (210)(729)x^6(16) + (252)(243)x^5(-32) + (210)(81)x^4(64) \\
 &+ (120)(27)x^3(-128) + (45)(9)x^2(256) + (10)(3)x(-512) + (1)(1)(1)(1024) \\
 &= 59049x^{10} - 393660x^9 + 1180980x^8 - 2099520x^7 + 2449440x^6 - 1959552x^5 \\
 &+ 1088640x^4 - 414720x^3 + 103680x^2 - 15360x + 1024
 \end{aligned}$$

- **Expand  $(2x - 5y)^7$**

I'll plug "2x", "-5y", and "7" into the Binomial Theorem, counting up from zero to seven to get each term. (I mustn't forget the "minus" sign that goes with the second term in the binomial.)

$$\begin{aligned}(2x - 5y)^7 &= {}_7C_0 (2x)^7(-5y)^0 + {}_7C_1 (2x)^6(-5y)^1 + {}_7C_2 (2x)^5(-5y)^2 \\ &+ {}_7C_3 (2x)^4(-5y)^3 + {}_7C_4 (2x)^3(-5y)^4 + {}_7C_5 (2x)^2(-5y)^5 \\ &+ {}_7C_6 (2x)^1(-5y)^6 + {}_7C_7 (2x)^0(-5y)^7\end{aligned}$$

Then simplifying gives me:

$$\begin{aligned}(1)(128x^7)(1) &+ (7)(64x^6)(-5y) + (21)(32x^5)(25y^2) + (35)(16x^4)(-125y^3) \\ &+ (35)(8x^3)(625y^4) + (21)(4x^2)(-3125y^5) + (7)(2x)(15625y^6) \\ &+ (1)(1)(-78125y^7) \\ &= 128x^7 - 2240x^6y + 16800x^5y^2 - 70000x^4y^3 + 175000x^3y^4 - 262500x^2y^5 \\ &\quad + 218750xy^6 - 78125y^7\end{aligned}$$

You may be asked to find a certain term in an expansion, the idea being that the exercise will be way easy if you've memorized the Theorem, but will be difficult or impossible if you haven't. So memorize the Theorem and get the easy points.

- **What is the fourth term in the expansion of  $(3x - 2)^{10}$ ?**

I've already expanded this binomial, so let's take a look:

$$\begin{aligned}(3x - 2)^{10} &= {}_{10}C_0 (3x)^{10-0}(-2)^0 + {}_{10}C_1 (3x)^{10-1}(-2)^1 + {}_{10}C_2 (3x)^{10-2}(-2)^2 \\ &+ {}_{10}C_3 (3x)^{10-3}(-2)^3 + {}_{10}C_4 (3x)^{10-4}(-2)^4 + {}_{10}C_5 (3x)^{10-5}(-2)^5 \\ &+ {}_{10}C_6 (3x)^{10-6}(-2)^6 + {}_{10}C_7 (3x)^{10-7}(-2)^7 + {}_{10}C_8 (3x)^{10-8}(-2)^8 \\ &+ {}_{10}C_9 (3x)^{10-9}(-2)^9 + {}_{10}C_{10} (3x)^{10-10}(-2)^{10}\end{aligned}$$

So the fourth term is not the one where I've counted up to 4, but the one where I've counted up just to 3. (This is because, just as with Javascript, the counting starts with 0, not 1.)

Note that, in any expansion, there is one more term than the number in the power. For instance:

$$(x + y)^2 = x^2 + 2xy + y^2 \quad (\text{second power: three terms})$$

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 \quad (\text{third power: four terms})$$

$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 \quad (\text{fourth power: five terms})$$

The expansion in this exercise,  $(3x - 2)^{10}$ , has power of  $n = 10$ , so the expansion will have eleven terms, and the terms will count up, not from 1 to 10 or from 1 to 11, but from 0 to 10. This is why the fourth term will not be the one where I'm using "4" as my counter, but will be the one where I'm using "3".

$${}_{10}C_3 (3x)^{10-3}(-2)^3 = (120)(2187)(x^7)(-8) = -2099520x^7$$

### **Practice problems**

1) Expand  $(2x + 5)^7$

2) Expand  $(x^2 + 3)^6$

3) Find the 10<sup>th</sup> term in the expansion of  $(x + 3)^{12}$

4) Find the 5<sup>th</sup> term in the expansion of  $(4x - y)^8$