57. $\sin\theta = 0.9652$ with θ in QII
58. $\sin\theta = 0.9652$ with θ in QI
59. $\sec\theta = 1.4325$ with θ in QIV
60. $\csc\theta = 1.4325$ with θ in QII
61. $\csc\theta = 2.4957$ with θ in QII
62. $\sec\theta = -3.4159$ with θ in QII
63. $\cot\theta = -0.7366$ with θ in QII
64. $\cot\theta = -0.1234$ with θ in QIV
65. $\sec\theta = -1.7876$ with θ in QIII
66. $\csc\theta = -1.7876$ with θ in QIII

Find θ, if $0° < \theta < 360°$ and

67. $\sin\theta = -\frac{\sqrt{3}}{2}$ and θ in QIII
68. $\sin\theta = -\frac{1}{\sqrt{2}}$ and θ in QIII
69. $\cos\theta = -\frac{1}{\sqrt{2}}$ and θ in QII
70. $\cos\theta = -\frac{\sqrt{3}}{2}$ and θ in QIII
71. $\sin\theta = -\frac{\sqrt{3}}{2}$ and θ in QIV
72. $\sin\theta = \frac{1}{\sqrt{2}}$ and θ in QII
73. $\tan\theta = \sqrt{3}$ and θ in QIII
74. $\tan\theta = \frac{1}{\sqrt{3}}$ and θ in QIII
75. $\sec\theta = -2$ with θ in QII
76. $\csc\theta = 2$ with θ in QII
77. $\csc\theta = \sqrt{2}$ with θ in QII
78. $\sec\theta = \sqrt{2}$ with θ in QIV
79. $\cot\theta = -1$ with θ in QIV
80. $\cot\theta = \sqrt{3}$ with θ in QIII

REVIEW PROBLEMS

The problems that follow review material we covered in Sections 1.1 and 2.1. Give the complement and supplement of each angle.

81. $70°$
82. $120°$
83. x
84. $90° - y$

85. If the longest side in a 30°–60°–90° triangle is 10, find the length of the other two sides.
86. If the two shorter sides of a 45°–45°–90° triangle are both $\frac{3}{4}$, find the length of the hypotenuse.

Simplify each expression by substituting values from the table of exact values and then simplifying the resulting expression.

87. $\sin 30° \cos 60°$
88. $4\sin 60° - 2\cos 30°$
89. $\sin^2 45° + \cos^2 45°$
90. $(\sin 45° + \cos 45°)^2$

SECTION 3.2 RADIANS AND DEGREES

If you think back to the work you have done with functions of the form $y = f(x)$ in your algebra class, you will see that the variables x and y were always real numbers. The trigonometric functions we have worked with so far have had the form $y = f(\theta)$, where θ is measured in degrees. In order to apply the knowledge we have about functions from algebra to our trigonometric functions, we need to write our angles as real numbers, not degrees. The key to doing this is called *radian measure.* Radian measure is a relatively new concept in the history of mathematics. The term *radian* was first introduced by physicist James T. Thomson in examination questions in 1873. The introduction of radian measure will allow us to do a number of useful things. For instance, in Chapter 4 we will graph the function $y = \sin x$ on a rectan-

Although James Thomson is credited with the first printed use of the term *radian,* it is believed that the concept of radian measure was originally proposed by Roger Cotes (1682–1716), who was also the first to calculate 1 radian in degrees.

gular coordinate system, where the units on the x and y axes are given by real numbers, just as they would be if we were graphing $y = 2x + 3$ or $y = x^2$. To understand the definition for radian measure, we have to recall from geometry that a central angle in a circle is an angle with its vertex at the center of the circle. Here is the definition for an angle with a measure of 1 radian.

DEFINITION

In a circle, a central angle that cuts off an arc equal in length to the radius of the circle has a measure of 1 *radian* (rad). Figure 1 illustrates this.

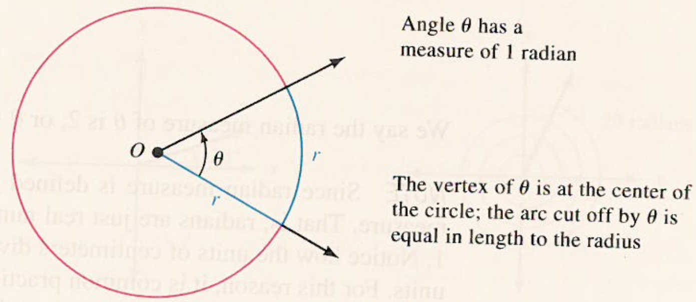

Figure 1

To find the radian measure of *any* central angle, we must find how many radii are in the arc it cuts off. To do so, we divide the arc length by the radius. If the radius is 2 centimeters and the arc cut off by central angle θ is 6 centimeters, then the radian measure of θ is $\frac{6}{2} = 3$ rad. Here is the formal definition:

DEFINITION RADIAN MEASURE

If a central angle θ, in a circle of radius r, cuts off an arc of length s, then the measure of θ, in radians, is given by s/r (Figure 2).

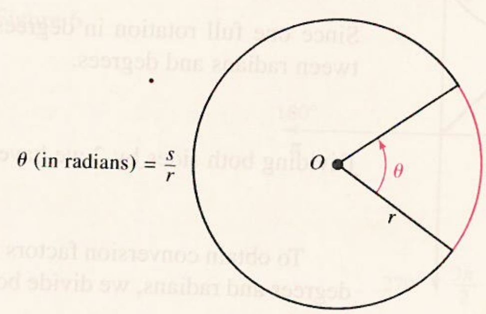

Figure 2

As you will see later in this section, one radian is equal to approximately 57.3°.

EXAMPLE 1 A central angle θ in a circle of radius 3 centimeters cuts off an arc of length 6 centimeters What is the radian measure of θ?

SOLUTION We have $r = 3$ cm and $s = 6$ cm (Figure 3); therefore,

$$\theta \text{ (in radians)} = \frac{s}{r} = \frac{6 \text{ cm}}{3 \text{ cm}} = 2$$

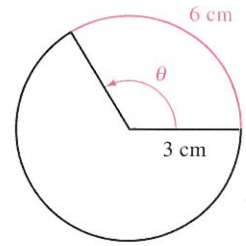

Figure 3

We say the radian measure of θ is 2, or $\theta = 2$ rad.

NOTE Since radian measure is defined as a ratio, s/r, technically it is a unitless measure. That is, radians are just real numbers. To see why, look again at Example 1. Notice how the units of centimeters divide out, leaving the number 2 without any units. For this reason, it is common practice to omit the word *radian* when using radian measure. (To avoid confusion, we will sometimes use the term *rad* in this book as if it were a unit.) If no units are showing, an angle is understood to be measured in radians; with degree measure, the degree symbol ° must be written.

$\theta = 2$ means the measure of θ is 2 radians

$\theta = 2°$ means the measure of θ is 2 degrees

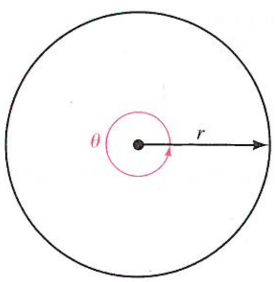

Figure 4

To see the relationship between degrees and radians, we can compare the number of degrees and the number of radians in one full rotation (Figure 4).

The angle formed by one full rotation about the center of a circle of radius r will cut off an arc equal to the circumference of the circle. Since the circumference of a circle of radius r is $2\pi r$, we have

θ measures one full rotation	$\theta = \dfrac{2\pi r}{r} = 2\pi$	The measure of θ in radians is 2π

Since one full rotation in degrees is 360°, we have the following relationship between radians and degrees.

$$360° = 2\pi \text{ rad}$$

Dividing both sides by 2 we have

$$180° = \pi \text{ rad}$$

To obtain conversion factors that will allow us to change back and forth between degrees and radians, we divide both sides of this last equation alternately by 180 and by π.

Divide both sides by 180 ← $180° = \pi$ rad → Divide both sides by π

$$1° = \frac{\pi}{180} \text{ rad} \qquad \left(\frac{180}{\pi}\right)^\circ = 1 \text{ rad}$$

To gain some insight into the relationship between degrees and radians, we can approximate π with 3.14 to obtain the approximate number of degrees in 1 radian.

$$\begin{aligned} 1 \text{ rad} &= 1\left(\frac{180}{\pi}\right)^\circ \\ &\approx 1\left(\frac{180}{3.14}\right)^\circ \\ &= 57.3^\circ \qquad \text{To the nearest tenth} \end{aligned}$$

We see that 1 radian is approximately 57°. A radian is much larger than a degree. Figure 5 illustrates the relationship between 20° and 20 radians.

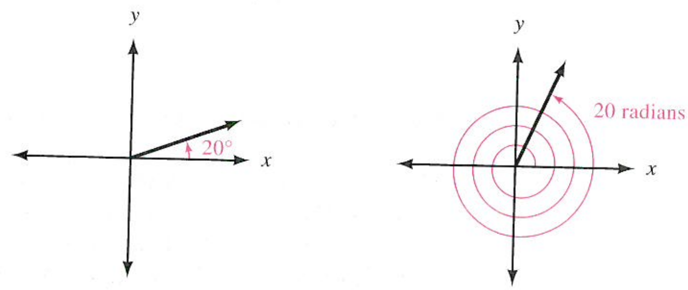

Figure 5

Here are some further conversions between degrees and radians.

CONVERTING FROM DEGREES TO RADIANS

EXAMPLE 2 Convert 45° to radians.

SOLUTION Since $1^\circ = \frac{\pi}{180}$ radians, and 45° is the same as 45(1°), we have

$$45^\circ = 45\left(\frac{\pi}{180}\right) \text{ rad} = \frac{\pi}{4} \text{ rad}$$

as illustrated in Figure 6.

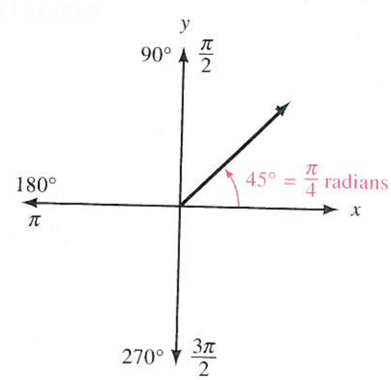

Figure 6

When we have our answer in terms of π, as in $\pi/4$, we are writing an exact value. If we wanted a decimal approximation, we would substitute 3.14 for π.

$$\text{Exact value} \quad \frac{\pi}{4} \approx \frac{3.14}{4} = 0.785 \quad \text{Approximate value}$$

Note that if we wanted the radian equivalent of 90°, we could simply multiply $\pi/4$ by 2, since $90° = 2 \times 45°$.

$$90° = 2 \times 45° = 2 \times \frac{\pi}{4} = \frac{\pi}{2}$$

EXAMPLE 3 Convert 450° to radians.

SOLUTION As illustrated in Figure 7, multiplying by $\pi/180$ we have

$$450° = 450\left(\frac{\pi}{180}\right) = \frac{5\pi}{2} \text{ rad}$$

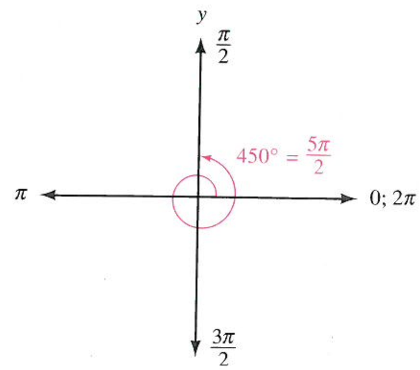

Figure 7

Again, $5\pi/2$ is the exact value. If we wanted a decimal approximation, we would substitute 3.14 for π.

$$\text{Exact value} \quad \frac{5\pi}{2} \approx \frac{5(3.14)}{2} = 7.85 \quad \text{Approximate value}$$

CONVERTING FROM RADIANS TO DEGREES

EXAMPLE 4 Convert $\pi/6$ to degrees.

SOLUTION To convert from radians to degrees, we multiply by $180/\pi$.

$$\frac{\pi}{6} \text{ (rad)} = \frac{\pi}{6}\left(\frac{180}{\pi}\right)^\circ = 30°$$

Note that 60° is twice 30°, so $2(\pi/6) = \pi/3$ must be the radian equivalent of 60°. Figure 8 illustrates.

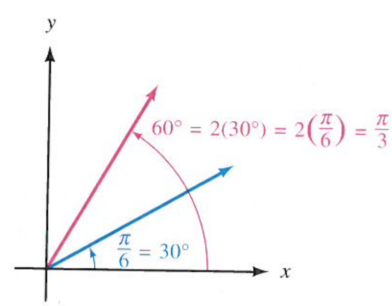

Figure 8

EXAMPLE 5 Convert $4\pi/3$ to degrees.

SOLUTION Multiplying by $180/\pi$ we have

$$\frac{4\pi}{3}\text{ (rad)} = \frac{4\pi}{3}\left(\frac{180}{\pi}\right)^{\circ} = 240^{\circ}$$

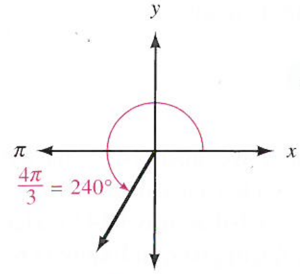

Figure 9

Note that the reference angle for the angle shown in Figure 9 can be given in either degrees or radians.

$$\text{In degrees: } \hat{\theta} = 240^{\circ} - 180^{\circ} = 60^{\circ}$$

$$\text{In radians: } \hat{\theta} = \frac{4\pi}{3} - \pi = \frac{4\pi}{3} - \frac{3\pi}{3} = \frac{\pi}{3}$$

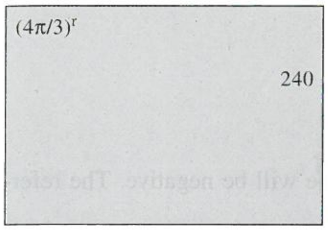

Figure 10

CALCULATOR NOTE Some calculators have the ability to convert angles from degree measure to radian measure, and vice versa. For example, Figure 10 shows how Example 5 could be done on a graphing calculator that is set to degree mode. Consult your calculator manual to see if your model is able to perform angle conversions.

As is apparent from the preceding examples, changing from degrees to radians and radians to degrees is simply a matter of multiplying by the appropriate conversion factors.

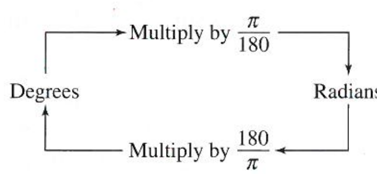

Table 1 displays the conversions between degrees and radians for the special angles, and also summarizes the exact values of the sine, cosine, and tangent of these angles for your convenience.

TABLE 1

θ		Value of Trigonometric Function		
Degrees	*Radians*	$\sin\theta$	$\cos\theta$	$\tan\theta$
0°	0	0	1	0
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	$\frac{\pi}{2}$	1	0	Undefined
180°	π	0	−1	0
270°	$\frac{3\pi}{2}$	−1	0	Undefined
360°	2π	0	1	0

EXAMPLE 6 Find $\sin \frac{\pi}{6}$.

SOLUTION Since $\pi/6$ and 30° are equivalent, so are their sines.

$$\sin \frac{\pi}{6} = \sin 30° = \frac{1}{2}$$

CALCULATOR NOTE To work this problem on a calculator, we must first set the calculator to radian mode. (Consult the manual that came with your calculator to see how to do this.) If your calculator does not have a key labeled π, use 3.1416. Here is the sequence to key in your calculator to work the problem given in Example 6.

Scientific Calculator	Graphing Calculator
3.1416 [÷] 6 [=] [sin]	[sin] [(] [π] [÷] 6 [)] [ENTER]

EXAMPLE 7 Find $4 \sin \frac{7\pi}{6}$.

SOLUTION Since $7\pi/6$ terminates in QIII, its sine will be negative. The reference angle is $7\pi/6 - \pi = \pi/6$ (Figure 11).

$$\begin{aligned} 4 \sin \frac{7\pi}{6} &= 4\left(-\sin \frac{\pi}{6}\right) \\ &= 4\left(-\frac{1}{2}\right) \\ &= -2 \end{aligned}$$

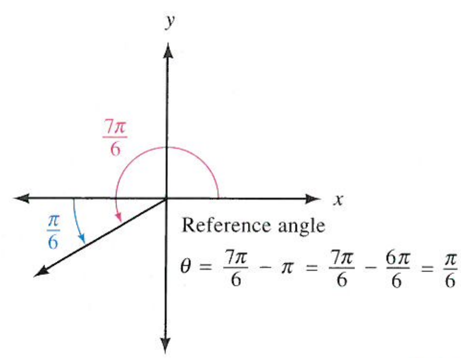

Figure 11

EXAMPLE 8 Evaluate $4 \sin (2x + \pi)$ when $x = \pi/6$.

SOLUTION Substituting $\pi/6$ for x and simplifying, we have

$$\begin{aligned} 4 \sin\left(2 \cdot \frac{\pi}{6} + \pi\right) &= 4 \sin\left(\frac{\pi}{3} + \pi\right) \\ &= 4 \sin \frac{4\pi}{3} \\ &= 4\left(-\sin \frac{\pi}{3}\right) \\ &= 4\left(-\frac{\sqrt{3}}{2}\right) \\ &= -2\sqrt{3} \end{aligned}$$

GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

a. What is a radian?

b. What is radian measure?

c. Explain how to convert from radian measure to degree measure.

d. Explain how to convert from degree measure to radian measure.

PROBLEM SET 3.2

Find the radian measure of angle θ, if θ is a central angle in a circle of radius r, and θ cuts off an arc of length s.

1. $r = 3$ cm, $s = 9$ cm

2. $r = 6$ cm, $s = 3$ cm

3. $r = 10$ inches, $s = 5$ inches

4. $r = 5$ inches, $s = 10$ inches

5. $r = 4$ inches, $s = 12\pi$ inches

6. $r = 3$ inches, $s = 12$ inches

7. $r = \dfrac{1}{4}$ cm, $s = \dfrac{1}{2}$ cm

8. $r = \dfrac{1}{4}$ cm, $s = \dfrac{1}{8}$ cm

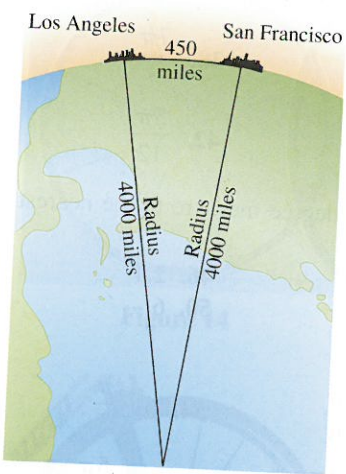

Figure 12

9. Angle Between Cities Los Angeles and San Francisco are approximately 450 miles apart on the surface of the earth. Assuming that the radius of the earth is 4,000 miles, find the radian measure of the central angle with its vertex at the center of the earth that has Los Angeles on one side and San Francisco on the other side (Figure 12).

10. Angle Between Cities Los Angeles and New York City are approximately 2,500 miles apart on the surface of the earth. Assuming that the radius of the earth is 4,000 miles, find the radian measure of the central angle with its vertex at the center of the earth that has Los Angeles on one side and New York City on the other side.

For each angle below:

a. Draw the angle in standard position.

b. Convert to radian measure using exact values.

c. Name the reference angle in both degrees and radians.

11. $30°$

12. $60°$

13. $90°$

14. $270°$

15. $260°$

16. $340°$

17. $-150°$

18. $-210°$

19. $420°$

20. $390°$

21. $-135°$

22. $-120°$

For Problems 23–26, use 3.1416 for π unless your calculator has a key marked π.

23. Use a calculator to convert $120° 40'$ to radians. Round your answer to the nearest hundredth. (First convert to decimal degrees, then multiply by the appropriate conversion factor to convert to radians.)

24. Use a calculator to convert $256° 20'$ to radians to the nearest hundredth of a radian.

25. Use a calculator to convert $1'$ (1 min) to radians to three significant digits.

26. Use a calculator to convert $1°$ to radians to three significant digits.

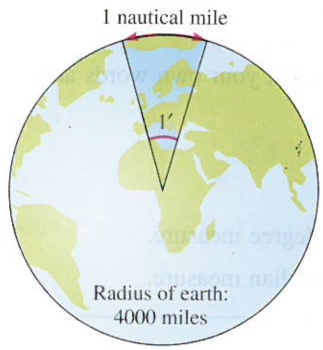

Figure 13

27. Nautical Miles If a central angle with its vertex at the center of the earth has a measure of 1′, then the arc on the surface of the earth that is cut off by this angle has a measure of 1 nautical mile (Figure 13). Find the number of regular (statute) miles in 1 nautical mile to the nearest hundredth of a mile. (Use 4,000 miles for the radius of the earth.)

28. Nautical Miles If two ships are 20 nautical miles apart on the ocean, how many statute miles apart are they? (Use the result of Problem 27 to do the calculations.)

29. Clock Through how many radians does the minute hand of a clock turn during a 5-minute period?

30. Clock Through how many radians does the minute hand of a clock turn during a 25-minute period?

For each angle below:

a. Convert to degree measure.
b. Draw the angle in standard position.
c. Label the reference angle in both degrees and radians.

31. $\frac{\pi}{3}$ **32.** $\frac{\pi}{4}$ **33.** $\frac{2\pi}{3}$ **34.** $\frac{3\pi}{4}$

35. $\frac{-7\pi}{6}$ **36.** $\frac{-5\pi}{6}$ **37.** $\frac{5\pi}{3}$ **38.** $\frac{7\pi}{3}$

39. 4π **40.** 3π **41.** $\frac{\pi}{12}$ **42.** $\frac{5\pi}{12}$

Use a calculator to convert each of the following to degree measure to the nearest tenth of a degree.

43. 1 **44.** 2 **45.** 1.3 **46.** 2.4
47. 0.75 **48.** 0.25 **49.** 5 **50.** 6

Give the exact value of each of the following:

51. $\sin \frac{4\pi}{3}$ **52.** $\cos \frac{4\pi}{3}$ **53.** $\tan \frac{\pi}{6}$ **54.** $\cot \frac{\pi}{3}$

55. $\sec \frac{2\pi}{3}$ **56.** $\csc \frac{3\pi}{2}$ **57.** $\csc \frac{5\pi}{6}$ **58.** $\sec \frac{5\pi}{6}$

59. $4 \sin\left(-\frac{\pi}{4}\right)$ **60.** $4 \cos\left(-\frac{\pi}{4}\right)$

61. $-\sin \frac{\pi}{4}$ **62.** $-\cos \frac{\pi}{4}$ **63.** $2 \cos \frac{\pi}{6}$ **64.** $2 \sin \frac{\pi}{6}$

Evaluate each of the following expressions when x is $\pi/6$. In each case, use exact values.

65. $\sin 2x$ **66.** $\sin 3x$ **67.** $6 \cos 3x$ **68.** $6 \cos 2x$

69. $\sin\left(x + \frac{\pi}{2}\right)$ **70.** $\sin\left(x - \frac{\pi}{2}\right)$

71. $4 \cos\left(2x + \frac{\pi}{3}\right)$ **72.** $4 \cos\left(3x + \frac{\pi}{6}\right)$

For the following expressions, find the value of y that corresponds to each value of x, then write your results as ordered pairs (x, y).

73. $y = \sin x$ for $x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$

74. $y = \cos x$ for $x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$

75. $y = 2\sin x$ for $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

76. $y = \frac{1}{2}\cos x$ for $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

77. $y = \sin 2x$ for $x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$

78. $y = \cos 3x$ for $x = 0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}$

79. $y = \sin\left(x - \frac{\pi}{2}\right)$ for $x = \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, \frac{5\pi}{2}$

80. $y = \cos\left(x - \frac{\pi}{6}\right)$ for $x = \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{7\pi}{6}$

81. $y = 3\sin\left(2x + \frac{\pi}{2}\right)$ for $x = -\frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$

82. $y = 5\cos\left(2x - \frac{\pi}{3}\right)$ for $x = \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{7\pi}{6}$

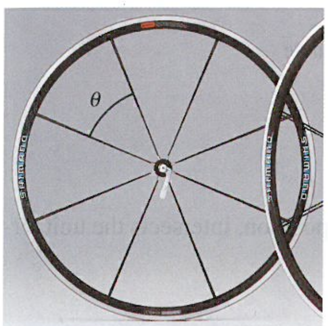

Figure 14

83. Cycling The Shimano WH-R540 aluminum wheel has 8 pairs of spokes evenly distributed around the rim of the wheel. What is the measure, in radians, of the central angle formed by adjacent pairs of spokes (Figure 14)?

Figure 15

84. Cycling The Mavic Ksyrium Elite wheel has 18 spokes evenly distributed around the rim of the wheel. What is the measure, in radians, of the central angle formed by adjacent spokes (Figure 15)?

REVIEW PROBLEMS

The problems that follow review material we covered in Section 1.3.

Find all six trigonometric functions of θ, if the given point is on the terminal side of θ.

85. $(1, -3)$ **86.** $(-1, 3)$ **87.** (m, n) **88.** (a, b)

89. Find the remaining trigonometric functions of θ, if $\sin\theta = \frac{1}{2}$ and θ terminates in QII.

90. Find the remaining trigonometric functions of θ, if $\cos\theta = -1/\sqrt{2}$ and θ terminates in QII.

91. Find all six trigonometric functions of θ, if the terminal side of θ lies along the line $y = 2x$ in QI.

92. Find the six trigonometric functions of θ, if the terminal side of θ lies along the line $y = 2x$ in QIII.