5.1 Basic Identities

TABLE 1

	Basic Identities	Common Equivalent Forms
Reciprocal	$\csc\theta = \dfrac{1}{\sin\theta}$	$\sin\theta = \dfrac{1}{\csc\theta}$
	$\sec\theta = \dfrac{1}{\cos\theta}$	$\cos\theta = \dfrac{1}{\sec\theta}$
	$\cot\theta = \dfrac{1}{\tan\theta}$	$\tan\theta = \dfrac{1}{\cot\theta}$
Ratio	$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$	
	$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$	
Pythagorean	$\cos^2\theta + \sin^2\theta = 1$	$\sin^2\theta = 1 - \cos^2\theta$ $\sin\theta = \pm\sqrt{1-\cos^2\theta}$ $\cos^2\theta = 1 - \sin^2\theta$ $\cos\theta = \pm\sqrt{1-\sin^2\theta}$
	$1 + \tan^2\theta = \sec^2\theta$	
	$1 + \cot^2\theta = \csc^2\theta$	

NOTE The last two Pythagorean identities can be derived from $\cos^2\theta + \sin^2\theta = 1$ by dividing each side by $\cos^2\theta$ and $\sin^2\theta$, respectively. For example, if we divide each side of $\cos^2\theta + \sin^2\theta = 1$ by $\cos^2\theta$, we have

$$\cos^2\theta + \sin^2\theta = 1$$

$$\frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$1 + \tan^2\theta = \sec^2\theta$$

To derive the last Pythagorean identity, we would need to divide both sides of $\cos^2\theta + \sin^2\theta = 1$ by $\sin^2\theta$ to obtain $1 + \cot^2\theta = \csc^2\theta$.

The rest of this section is concerned with using the basic identities (or their equivalent forms) listed above, along with our knowledge of algebra, to prove other identities.

Recall that an identity in trigonometry is a statement that two expressions are equal for all replacements of the variable for which each expression is defined. To prove (or verify) a trigonometric identity, we use trigonometric substitutions and algebraic manipulations to either

1. Transform the right side of the identity into the left side, or
2. Transform the left side of the identity into the right side.

The main thing to remember in proving identities is to work on each side of the identity separately. We do not want to use properties from algebra that involve both sides of the identity—like the addition property of equality. We prove identities in order to develop the ability to transform one trigonometric expression into another.

When we encounter problems in other courses that require the use of the techniques used to verify identities, we usually find that the solution to these problems hinges on transforming an expression containing trigonometric functions into less complicated expressions. In these cases, we do not usually have an equal sign to work with.

EXAMPLE 1 Prove $\sin\theta\cot\theta = \cos\theta$.

PROOF To prove this identity, we transform the left side into the right side.

$$\sin\theta\cot\theta = \sin\theta\cdot\frac{\cos\theta}{\sin\theta} \quad \text{Ratio identity}$$

$$= \frac{\sin\theta\cos\theta}{\sin\theta} \quad \text{Multiply}$$

$$= \cos\theta \quad \text{Divide out common factor } \sin\theta$$

In this example, we have transformed the left side into the right side. Remember, we verify identities by transforming one expression into another.

EXAMPLE 2 Prove $\tan x + \cos x = \sin x\,(\sec x + \cot x)$.

PROOF We can begin by applying the distributive property to the right side to multiply through by $\sin x$. Then we can change each expression on the right side to an equivalent expression involving only $\sin x$ and $\cos x$.

$$\sin x\,(\sec x + \cot x) = \sin x\sec x + \sin x\cot x \quad \text{Multiply}$$

$$= \sin x\cdot\frac{1}{\cos x} + \sin x\cdot\frac{\cos x}{\sin x} \quad \text{Reciprocal and ratio identities}$$

$$= \frac{\sin x}{\cos x} + \cos x \quad \text{Multiply}$$

$$= \tan x + \cos x \quad \text{Ratio identity}$$

In this case, we transformed the right side into the left side.

Before we go on to the next example, let's list some guidelines that may be useful in learning how to prove identities.

GUIDELINES FOR PROVING IDENTITIES

1. It is usually best to work on the more complicated side first.
2. Look for trigonometric substitutions involving the basic identities that may help simplify things.
3. Look for algebraic operations, such as adding fractions, the distributive property, or factoring, that may simplify the side you are working with or that will at least lead to an expression that will be easier to simplify.
4. If you cannot think of anything else to do, change everything to sines and cosines and see if that helps.
5. Always keep an eye on the side you are not working with to be sure you are working toward it. There is a certain sense of direction that accompanies a successful proof.

Probably the best advice is to remember that these are simply guidelines. The best way to become proficient at proving trigonometric identities is to practice. The more identities you prove, the more you will be able to prove and the more confident you will become. *Don't be afraid to stop and start over if you don't seem to be getting anywhere.* With most identities, there are a number of different proofs that will lead to the same result. Some of the proofs will be longer than others.

EXAMPLE 3 Prove $\dfrac{\cos^4 t - \sin^4 t}{\cos^2 t} = 1 - \tan^2 t$.

PROOF In this example, factoring the numerator on the left side will reduce the exponents there from 4 to 2.

$$\frac{\cos^4 t - \sin^4 t}{\cos^2 t} = \frac{(\cos^2 t + \sin^2 t)(\cos^2 t - \sin^2 t)}{\cos^2 t} \qquad \text{Factor}$$

$$= \frac{1\cdot(\cos^2 t - \sin^2 t)}{\cos^2 t} \qquad \text{Pythagorean identity}$$

$$= \frac{\cos^2 t}{\cos^2 t} - \frac{\sin^2 t}{\cos^2 t} \qquad \text{Separate into two fractions}$$

$$= 1 - \tan^2 t \qquad \text{Ratio identity}$$

EXAMPLE 4 Prove $1 + \cos\theta = \dfrac{\sin^2\theta}{1 - \cos\theta}$.

PROOF We begin this proof by applying an alternate form of the Pythagorean identity to the right side to write $\sin^2\theta$ as $1 - \cos^2\theta$. Then we factor $1 - \cos^2\theta$ as the difference of two squares and reduce to lowest terms.

$$\frac{\sin^2\theta}{1 - \cos\theta} = \frac{1 - \cos^2\theta}{1 - \cos\theta} \qquad \text{Pythagorean identity}$$

$$= \frac{(1 - \cos\theta)(1 + \cos\theta)}{1 - \cos\theta} \qquad \text{Factor}$$

$$= 1 + \cos\theta \qquad \text{Reduce}$$

USING TECHNOLOGY

VERIFYING IDENTITIES

You can use your graphing calculator to decide if an equation is an identity or not. If the two expressions are indeed equal for all defined values of the variable, then they should produce identical graphs. Although this does not constitute a proof, it does give strong evidence that the identity is true.

We can verify the identity in Example 4 by defining the expression on the left as a function Y_1 and the expression on the right as a second function Y_2. If your calculator is equipped with different graphing styles, set the style of Y_2 so that you will be able to distinguish the second graph from the first. (In Figure 1, we have used the *path* style on a TI-83 for the second function.) Also, be sure your calculator is set to radian mode. Set your window variables so that

```
Plot1 Plot2 Plot3
\Y1=1+cos(X)
-0Y2=(sin(X))²/(1-cos(X))
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

Figure 1

$$-2\pi \le x \le 2\pi, \text{ scale} = \pi/2; \; -4 \le y \le 4, \text{ scale} = 1$$

When you graph the functions, your calculator screen should look similar to Figure 2 (the small circle is a result of the path style in action). Observe that the two graphs are identical.

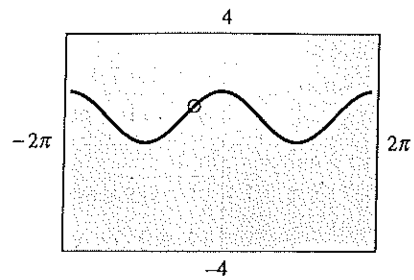

Figure 2

If your calculator is not equipped with different graphing styles, it may be difficult to tell if the second graph really coincides with the first. In this case you can trace the graph, and switch between the two functions at several points to convince yourself that the two graphs are indeed the same.

EXAMPLE 5 Prove $\tan x + \cot x = \sec x \csc x$.

PROOF We begin this proof by writing the left side in terms of $\sin x$ and $\cos x$. Then we simplify the left side by finding a common denominator in order to add the resulting fractions.

$$\begin{aligned}
\tan x + \cot x &= \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} && \text{Changes to sines and cosines} \\
&= \frac{\sin x}{\cos x} \cdot \frac{\mathbf{\sin x}}{\mathbf{\sin x}} + \frac{\cos x}{\sin x} \cdot \frac{\mathbf{\cos x}}{\mathbf{\cos x}} && \text{LCD} \\
&= \frac{\sin^2 x + \cos^2 x}{\cos x \sin x} && \text{Add fractions} \\
&= \frac{1}{\cos x \sin x} && \text{Pythagorean identity} \\
&= \frac{1}{\cos x} \cdot \frac{1}{\sin x} && \text{Write as separate fractions} \\
&= \sec x \csc x && \text{Reciprocal identities}
\end{aligned}$$

EXAMPLE 6 Prove $\dfrac{\sin \alpha}{1 + \cos \alpha} + \dfrac{1 + \cos \alpha}{\sin \alpha} = 2 \csc \alpha$.

PROOF The common denominator for the left side of the equation is $\sin \alpha\,(1 + \cos \alpha)$. We multiply the first fraction by $(\sin \alpha)/(\sin \alpha)$ and the second fraction by $(1 + \cos \alpha)/(1 + \cos \alpha)$ to produce two equivalent fractions with the same denominator.

$$\frac{\sin\alpha}{1+\cos\alpha} + \frac{1+\cos\alpha}{\sin\alpha}$$

$$= \frac{\sin\alpha}{\sin\alpha} \cdot \frac{\sin\alpha}{1+\cos\alpha} + \frac{1+\cos\alpha}{\sin\alpha} \cdot \frac{1+\cos\alpha}{1+\cos\alpha}$$ LCD

$$= \frac{\sin^2\alpha + (1+\cos\alpha)^2}{\sin\alpha\,(1+\cos\alpha)}$$ Add numerators

$$= \frac{\sin^2\alpha + 1 + 2\cos\alpha + \cos^2\alpha}{\sin\alpha\,(1+\cos\alpha)}$$ Expand $(1+\cos\alpha)^2$

$$= \frac{2 + 2\cos\alpha}{\sin\alpha\,(1+\cos\alpha)}$$ Pythagorean identity

$$= \frac{2(1+\cos\alpha)}{\sin\alpha\,(1+\cos\alpha)}$$ Factor out a 2

$$= \frac{2}{\sin\alpha}$$ Reduce

$$= 2\csc\alpha$$ Reciprocal identity ▢

EXAMPLE 7 Prove $\dfrac{1+\sin t}{\cos t} = \dfrac{\cos t}{1-\sin t}$.

PROOF The trick to proving this identity requires that we multiply the numerator and denominator on the right side by $1 + \sin t$. (This is similar to rationalizing the denominator.)

$$\frac{\cos t}{1-\sin t} = \frac{\cos t}{1-\sin t} \cdot \frac{1+\sin t}{1+\sin t}$$ Multiply numerator and denominator by $1 + \sin t$

$$= \frac{\cos t\,(1+\sin t)}{1-\sin^2 t}$$ Multiply out the denominator

$$= \frac{\cos t\,(1+\sin t)}{\cos^2 t}$$ Pythagorean identity

$$= \frac{1+\sin t}{\cos t}$$ Reduce

Note that it would have been just as easy for us to verify this identity by multiplying the numerator and denominator on the left side by $1 - \sin t$. ▢

GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

a. What is an identity?

b. In trigonometry, how do we prove an identity?

c. What is a first step in simplifying the expression $\dfrac{\cos^4 t - \sin^4 t}{\cos^2 t}$?

d. What is a first step in simplifying the expression $\dfrac{\sin\alpha}{1+\cos\alpha} + \dfrac{1+\cos\alpha}{\sin\alpha}$?

PROBLEM SET 5.1

Prove that each of the following identities is true:

1. $\cos\theta \tan\theta = \sin\theta$
2. $\sec\theta \cot\theta = \csc\theta$
3. $\csc\theta \tan\theta = \sec\theta$
4. $\tan\theta \cot\theta = 1$
5. $\dfrac{\tan A}{\sec A} = \sin A$
6. $\dfrac{\cot A}{\csc A} = \cos A$
7. $\sec\theta \cot\theta \sin\theta = 1$
8. $\tan\theta \csc\theta \cos\theta = 1$
9. $\cos x(\csc x + \tan x) = \cot x + \sin x$
10. $\sin x(\sec x + \csc x) = \tan x + 1$
11. $\cot x - 1 = \cos x(\csc x - \sec x)$
12. $\tan x(\cos x + \cot x) = \sin x + 1$
13. $\cos^2 x(1 + \tan^2 x) = 1$
14. $\sin^2 x(\cot^2 x + 1) = 1$
15. $(1 - \sin x)(1 + \sin x) = \cos^2 x$
16. $(1 - \cos x)(1 + \cos x) = \sin^2 x$
17. $\dfrac{\cos^4 t - \sin^4 t}{\sin^2 t} = \cot^2 t - 1$
18. $\dfrac{\sin^4 t - \cos^4 t}{\sin^2 t \cos^2 t} = \sec^2 t - \csc^2 t$
19. $1 + \sin\theta = \dfrac{\cos^2\theta}{1 - \sin\theta}$
20. $1 - \sin\theta = \dfrac{\cos^2\theta}{1 + \sin\theta}$
21. $\dfrac{1 - \sin^4\theta}{1 + \sin^2\theta} = \cos^2\theta$
22. $\dfrac{1 - \cos^4\theta}{1 + \cos^2\theta} = \sin^2\theta$
23. $\sec^2\theta - \tan^2\theta = 1$
24. $\csc^2\theta - \cot^2\theta = 1$
25. $\sec^4\theta - \tan^4\theta = \dfrac{1 + \sin^2\theta}{\cos^2\theta}$
26. $\csc^4\theta - \cot^4\theta = \dfrac{1 + \cos^2\theta}{\sin^2\theta}$
27. $\tan\theta - \cot\theta = \dfrac{\sin^2\theta - \cos^2\theta}{\sin\theta\cos\theta}$
28. $\sec\theta - \csc\theta = \dfrac{\sin\theta - \cos\theta}{\sin\theta\cos\theta}$
29. $\csc B - \sin B = \cot B \cos B$
30. $\sec B - \cos B = \tan B \sin B$
31. $\cot\theta\cos\theta + \sin\theta = \csc\theta$
32. $\tan\theta\sin\theta + \cos\theta = \sec\theta$
33. $\dfrac{\cos x}{1 + \sin x} + \dfrac{1 + \sin x}{\cos x} = 2\sec x$
34. $\dfrac{\cos x}{1 + \sin x} - \dfrac{1 - \sin x}{\cos x} = 0$
35. $\dfrac{1}{1 + \cos x} + \dfrac{1}{1 - \cos x} = 2\csc^2 x$
36. $\dfrac{1}{1 - \sin x} + \dfrac{1}{1 + \sin x} = 2\sec^2 x$
37. $\dfrac{1 - \sec x}{1 + \sec x} = \dfrac{\cos x - 1}{\cos x + 1}$
38. $\dfrac{\csc x - 1}{\csc x + 1} = \dfrac{1 - \sin x}{1 + \sin x}$
39. $\dfrac{\cos t}{1 + \sin t} = \dfrac{1 - \sin t}{\cos t}$
40. $\dfrac{\sin t}{1 + \cos t} = \dfrac{1 - \cos t}{\sin t}$
41. $\dfrac{(1 - \sin t)^2}{\cos^2 t} = \dfrac{1 - \sin t}{1 + \sin t}$
42. $\dfrac{\sin^2 t}{(1 - \cos t)^2} = \dfrac{1 + \cos t}{1 - \cos t}$
43. $\dfrac{\sec\theta + 1}{\tan\theta} = \dfrac{\tan\theta}{\sec\theta - 1}$
44. $\dfrac{\csc\theta - 1}{\cot\theta} = \dfrac{\cot\theta}{\csc\theta + 1}$
45. $\dfrac{1 - \sin x}{1 + \sin x} = (\sec x - \tan x)^2$
46. $\dfrac{1 + \cos x}{1 - \cos x} = (\csc x + \cot x)^2$
47. $\sec x + \tan x = \dfrac{1}{\sec x - \tan x}$
48. $\dfrac{1}{\csc x - \cot x} = \csc x + \cot x$
49. $\dfrac{\sin x + 1}{\cos x + \cot x} = \tan x$
50. $\dfrac{\cos x + 1}{\cot x} = \sin x + \tan x$
51. $\sin^4 A - \cos^4 A = 1 - 2\cos^2 A$
52. $\cos^4 A - \sin^4 A = 1 - 2\sin^2 A$

53. $\dfrac{\sin^2 B - \tan^2 B}{1 - \sec^2 B} = \sin^2 B$

54. $\dfrac{\cot^2 B - \cos^2 B}{\csc^2 B - 1} = \cos^2 B$

55. $\dfrac{\sec^4 y - \tan^4 y}{\sec^2 y + \tan^2 y} = 1$

56. $\dfrac{\csc^2 y + \cot^2 y}{\csc^4 y - \cot^4 y} = 1$

57. $\dfrac{\sin^3 A - 8}{\sin A - 2} = \sin^2 A + 2\sin A + 4$

58. $\dfrac{1 - \cos^3 A}{1 - \cos A} = \cos^2 A + \cos A + 1$

59. $\dfrac{1 - \tan^3 t}{1 - \tan t} = \sec^2 t + \tan t$

60. $\dfrac{1 + \cot^3 t}{1 + \cot t} = \csc^2 t - \cot t$

61. $\dfrac{\tan x}{\sin x - \cos x} = \dfrac{\sin^2 x + \sin x \cos x}{\cos x - 2\cos^3 x}$

62. $\dfrac{\cot^2 x}{\sin x + \cos x} = \dfrac{\cos^2 x \sin x - \cos^3 x}{2\sin^4 x - \sin^2 x}$

The following identities are from the book *Plane and Spherical Trigonometry with Tables* by Rosenbach, Whitman, and Moskovitz, and published by Ginn and Company in 1937. Verify each identity.

63. $(\tan\theta + \cot\theta)^2 = \sec^2\theta + \csc^2\theta$

64. $\dfrac{\tan^2\psi + 2}{1 + \tan^2\psi} = 1 + \cos^2\psi$

65. $\dfrac{1 + \sin\phi}{1 - \sin\phi} - \dfrac{1 - \sin\phi}{1 + \sin\phi} = 4\tan\phi\sec\phi$

66. $\dfrac{\cos\beta}{1 - \tan\beta} + \dfrac{\sin\beta}{1 - \cot\beta} = \sin\beta + \cos\beta$

Use your graphing calculator to determine if each equation appears to be an identity or not by graphing the left expression and right expression together.

67. $(\sec B - 1)(\sec B + 1) = \tan^2 B$

68. $\dfrac{1 - \sec\theta}{\cos\theta} = \dfrac{\cos\theta}{1 + \sec\theta}$

69. $\sec x + \cos x = \tan x \sin x$

70. $\dfrac{\tan t}{\sec t + 1} = \dfrac{\sec t - 1}{\tan t}$

71. $\sec A - \csc A = \dfrac{\cos A - \sin A}{\cos A \sin A}$

72. $\cos^4\theta - \sin^4\theta = 2\cos^2\theta - 1$

73. $\dfrac{1}{1 - \sin x} + \dfrac{1}{1 + \sin x} = 2\sec^2 x$

74. $\cot^4 t - \tan^4 t = \dfrac{\sin^2 t + 1}{\cos^2 t}$

Show that each of the following statements is not an identity by finding a value of θ that makes the statement false.

75. $\sin\theta = \sqrt{1 - \cos^2\theta}$

76. $\sin\theta + \cos\theta = 1$

77. $\sin\theta = \dfrac{1}{\cos\theta}$

78. $\tan^2\theta + \cot^2\theta = 1$

79. $\sqrt{\sin^2\theta + \cos^2\theta} = \sin\theta + \cos\theta$

80. $\sin\theta\cos\theta = 1$